Some identities involving recursive sequences

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Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

- Introduction
- Period Fibonacci numbers
- Related problem
- Proof of the main theorem

1 Introduction

- The so-called Fibonacci zeta function and Lucas zeta function defined by

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \text{ and } \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s},$$

where the F_n and L_n denote the Fibonacci numbers and Lucas numbers.

- In 1980, Erdös proposed that if the following sums are irrational

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}} \text{ and } \sum_{n=0}^{\infty} \frac{1}{L_{2^n}},$$

which was proved by Andre-Jeannin and Badea.

And Is it true the following sum is irrational

$$\sum_{k=1}^{\infty} \frac{1}{F_{n_k}},$$

for $n_1 < n_2 \cdots$, with $\frac{n_{k+1}}{n_k} \geq c > 1$.

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

For the Fibonacci numbers $\{F_n\}$, and $\lfloor\cdot\rfloor$ denotes the floor function.

- **Theorem (Ohtsuka, Nakamura, 2009, The Fibonacci Quarterly)**

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{F_k}\right)^{-1}\right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2}-1, & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{F_k^2}\right)^{-1}\right\rfloor = \begin{cases} F_{n-1}F_n-1, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-1}F_n, & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

- **Theorem (Ohtsuka, 2011, The Fibonacci Quarterly)**

$$\left(\sum_{k=n}^{\infty}\frac{1}{F_k^2}\right)^{-1} = F_{n-1}F_n - \frac{(-1)^n}{3} + O\left(\frac{1}{F_n^2}\right).$$

$$\left(\sum_{k=n}^{\infty}\frac{1}{F_kF_{k+m}}\right)^{-1} = \sum_{k=1}^{n-1}\frac{1}{F_kF_{k+m}} + \frac{1}{3}F_{m-2(-1)^n} + O\left(\frac{1}{F_n^2}\right).$$

Problems

Let $\{u_n\}$ be the second-order linear recursive sequence, $u_n = au_{n-1} + bu_{n-2}$, where a and b are arbitrary reals, for any positive integer s and c.

Infinite sums

$$\left(\sum_{k=n}^{\infty} \frac{1}{u_k^s}\right)^{-1} = ?$$

Finite sums

$$\left(\sum_{k=n}^{cn} \frac{1}{u_k^s}\right)^{-1} = ?$$

Introduction

Periodic Fibonacci . . .

Related problem

Proof of the main theorem

- **Theorem (Zhang Wenpeng, Wang Tingting, 2012, Applied Mathematics and Computation)**

  For the Pell numbers $\{P_n\}$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_{n-1} + P_{n-2}, & \text{if } n \geq 2 \text{ is even;} \\ P_{n-1} + P_{n-2} - 1, & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k^2} \right)^{-1} \right\rfloor = \begin{cases} 2P_{n-1}P_n, & \text{if } n \geq 2 \text{ is even;} \\ 2P_{n-1}P_n, & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem
Home Page
Title Page
Page 6 of 31
Go Back
Full Screen
Close
Quit

- **Theorem (Xu Zhefeng, Wang Tingting, 2013, Advances in Difference Equations)**

  Let

$$P(3,n)=\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{P_k^3}\right)^{-1}\right\rfloor,$$

  then

$$P(3,n)=\begin{cases} P_n^2P_{n-1}+3P_nP_{n-1}^2+\left\lfloor -\frac{61}{82}P_n-\frac{91}{82}P_{n-1}\right\rfloor, & \text{if } n \text{ is even };\\ P_n^2P_{n-1}+3P_nP_{n-1}^2+\left\lfloor \frac{61}{82}P_n+\frac{91}{82}P_{n-1}\right\rfloor, & \text{if } n \text{ is odd }.\end{cases}$$

Introduction
Periodic Fibonacci...
Related problem
Proof of the main theorem

- **Theorem (Kilic, Arikan, 2013, Applied Mathematics and Computation)**

  For any positive integer $p \geq q$ and $n > k$,

$$u_n = pu_{n-1} + qu_{n-2} + u_{n-3} + \cdots + u_{n-k},$$

  there exists a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1}, \ (n \geq n_0),$$

  where $\| \cdot \|$ denotes the nearest integer. (Clearly, $\|x\| = \lfloor x + \frac{1}{2} \rfloor$.)

- **Definition** For any positive integer $n > m$, the mth-order linear recursive sequences $\{u_n\}$ is defined as follows

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_{m-1} u_{n-m+1} + a_m u_{n-m}, \tag{1}$$

  with initial values $u_i \in \mathbb{N}$ for $0 \leq i < m$ and at least one of them is not zero.

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

Theorem (Wu Zhengang, 2014, The Scientific World Journal)

Let $\{u_n\}$ be an mth-order sequence defined by (1) with the restriction $a_1, a_2, \cdots, a_m \in \mathbb{N}$ and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 2$. For any real number $\beta > 2$ and positive integer $1 \leq s < \lfloor \log_{\frac{\alpha}{a_1}} \alpha d \rfloor$, where $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ are the roots of the characteristic equation of u_n and $d^{-1} = \max\{|\alpha_1|, |\alpha_2|, \cdots, |\alpha_{m-1}|\}$, then there exists a positive integer n_2 such that

$$\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{a_1^{sk}}{u_k^s} \right)^{-1} - \left(\frac{u_n^s}{a_1^{sn}} - \frac{u_{n-1}^s}{a_1^{sn-s}} \right) \right\| = 0, (n \geq n_2).$$

Introduction

Periodic Fibonacci . . .

Related problem

Proof of the main theorem

2 Periodic Fibonacci numbers

- Edson introduced a new generalized Fibonacci sequence that depends on two real parameters as follows:

  Definition 1. For any positive integer a, b, the 2-periodic Fibonacci numbers is defined as follows.

$$u_n = \begin{cases} au_{n-1} + u_{n-2}, & \text{if } n \text{ is an even and } n \geq 2, \\ \\ bu_{n-1} + u_{n-2}, & \text{if } n \text{ is an odd and } n \geq 1, \end{cases}$$

  with initial values $u_0 = 0$ and $u_1 = 1$,

- **Generalized Binet's Formula** The terms of the generalized Fibonacci sequence $\{u_n\}$ are given by

$$u_n = \frac{a^{2\lfloor \frac{n}{2} \rfloor - n + 1}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \cdot \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

  where $\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}$, $\beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}$.

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

Theorem (Zhang Han, Wu Zhengang, 2013, Advances in Difference Equations)

(1)Let $\{u_n\}$ be a second-order sequence defined by Definition 1. For any even $p \geq 2$ and non-negative integer $q < p$, there exists a positive integer n_1 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_{pk+q}} \right)^{-1} \right\| = u_{pn+q} - u_{pn-p+q}$$

for all $n \geq n_1$.

(2) Let $\{u_n\}$ be a second-order sequence defined by Definition 1. For any integer $c \geq 0$, there exists a positive integer n_2 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} \right)^{-1} - \left(\frac{u_n u_{n+2c+1}}{a^n b^{n+2c+1}} - \frac{u_{n-1} u_{n+2c}}{a^{n-1} b^{n+2c}} \right) \right\| = 0$$

for all $n \geq n_2$.

Introduction
Periodic Fibonacci...
Related problem
Proof of the main theorem

- **Definition 2.** For any positive integer $x_1, x_2, \cdots, x_t$, let $q_0 = 0$, $q_1 = 1$, if $n \geq 2$, the sequence $\{q_n\}$ satisfies

$$q_n = \begin{cases} x_1 q_{n-1} + q_{n-2}, & \text{if } n \equiv 2 \mod t, \\ x_2 q_{n-1} + q_{n-2}, & \text{if } n \equiv 3 \mod t, \\ \vdots & \\ x_{t-1} q_{n-1} + q_{n-2}, & \text{if } n \equiv 0 \mod t, \\ x_t q_{n-1} + q_{n-2}, & \text{if } n \equiv 1 \mod t. \end{cases}$$

  We call $\{q_n\}$ the t-periodic Fibonacci numbers.

- **Generalized Binet's Formula.** The terms of the t-periodic Fibonacci numbers $\{q_n\}$ are given by

$$q_{st+r} = (-1)^{(t+1)s} \left(\left(\frac{\alpha_2^t - \beta_2^t}{\alpha_2 - \beta_2} \right) q_{s+r} - \left(\frac{\alpha_2^{t-1} - \beta_2^{t-1}}{\alpha_2 - \beta_2} \right) q_r \right),$$

  where

$$\alpha_2 = \frac{(-1)^t A + \sqrt{A^2 - (-1)^t 4}}{2}, \quad \beta_2 = \frac{(-1)^t A - \sqrt{A^2 - (-1)^t 4}}{2}.$$

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

Theorem. Let $\{q_n\}$ be the t-periodic Fibonacci numbers defined by Definition 2. For any positive integer s and a with

$$\left(\frac{A+\sqrt{A^2-4}}{2}\right)^{1-\frac{2}{ts}} < a < \left(\frac{A+\sqrt{A^2-4}}{2}\right)^{\frac{2}{ts}-1},$$

then there exists a positive integer n_2, for $n \geq n_2$,

$$\left\| \left(\sum_{k=n}^{\infty} \frac{a^{tks}}{\left(\prod_{i=1}^{t} q_{tk+i}\right)^s} \right)^{-1} - \left(\frac{\left(\prod_{i=1}^{t} q_{tn+i}\right)^s}{a^{tns}} - \frac{\left(\prod_{i=1}^{t} q_{tn-t+i}\right)^s}{a^{tns-ts}} \right) \right\| = 0.$$

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

3 Related problem

3.1. The binomial transform sequence

Theorem (Wu Zhengang, 2013, Advances in Difference Equations)

Let $\{X_n\}$ be a third-order linear recurrence sequence $X_{n+3} = a \cdot X_{n+2} + b \cdot X_{n+1} + c \cdot X_n$ with the initial values $X_0 = u$, $X_1 = v$ and $X_2 = w$ for all $n \geq 1$, where a, b and c are positive integers. For any positive integer $d \geq 2$, The binomial transform sequence T_n is defined by $\{X_{dn+1}\}$ as

$$T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot X_{dk+1}.$$

Then we have the recurrence formula

$$T_{n+1} = \frac{g_1 g_3 - g_1 g_6 + g_7}{g_3 - g_6} \cdot T_n + \frac{g_3 g_4 - g_2 g_6 - g_1 g_7}{g_3 - g_6} \cdot T_{n-1} + \frac{g_3 g_5 - g_2 g_7}{g_3 - g_6} \cdot T_{n-2}.$$

where

$$g_1 = f_1 + f_2 + c \cdot A_d f_5, \quad g_2 = f_3 A_{d+1} - f_1 f_2 + c \cdot A_d f_6,$$

$$g_3 = c \cdot A_{d+1}^2 - c \cdot A_d f_2 + c \cdot A_d f_4, \quad g_4 = f_3 A_{d+1} - f_1 f_2 + c \cdot A_d (f_5 + f_6),$$

$$g_5 = c \cdot A_d f_6, \quad g_6 = c \cdot (A_{d+1} - A_d f_2 + A_d f_4 - A_d), \quad g_7 = c \cdot A_d f_4,$$

and

$$f_1 = b \cdot A_d + c \cdot A_{d-1} + 1, \; f_2 = 1 + A_{d+2}, \; f_3 = b \cdot A_{d-1} + c \cdot A_d,$$

$$f_4 = 1 + c \cdot A_{d-1} - \frac{c \cdot A_d^2}{A_{d+1}}, \; f_5 = \frac{A_d}{A_{d+1}},$$

$$f_6 = b \cdot A_{d-1} + c \cdot A_{d-2} - \frac{b \cdot A_d^2 + c \cdot A_{d-1} A_d + A_d}{A_{d+1}},$$

the sequence $\{A_n\}$ is defined by $A_{n+3} = a \cdot A_{n+2} + b \cdot A_{n+1} + c \cdot A_n$ with the initial values $A_1 = 0$, $A_2 = 1$ and $A_3 = a$, for all $n \geq 1$.

Introduction
Periodic Fibonacci...
Related problem
Proof of the main theorem

Conjecture. For any positive integer $n > m$, the mth-order linear recursive sequences $\{X_n\}$ is defined as follows

$$X_n = a_1 X_{n-1} + a_2 X_{n-2} + \cdots + a_{m-1} X_{n-m+1} + a_m X_{n-m}, \tag{2}$$

with initial values $u_i \in \mathbb{N}$ for $0 \leq i < m$ and at least one of them is not zero. The binomial transform sequence T_n of $\{X_n\}$ is defined by

$$T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot X_{dk+1}.$$

For positive integer $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$, there exists a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{T_{pk+q}} \right)^{-1} \right\| = T_{pn+q} - T_{pn-p+q} \quad (n \geq n_0)$$

Introduction

Periodic Fibonacci . . .

Related problem

Proof of the main theorem

3.2. Other identities

Theorem (Zhang Wenpeng, 1996, The Fibonacci Quarterly)

Let $\{u_n\}$ be the second-order linear recursive sequence, $u_n = au_{n-1} + bu_{n-2}$, where a and b are arbitrary reals

$$\sum_{a_1+\cdots+a_k=n} U_{a_1}U_{a_2}\cdots U_{a_k} = \frac{U_1^{k-1}}{(b^2+4a)^{k-1}(k-1)!}[g_{k-1}(n)U_{n-k+1}+h_{k-1}(n)U_{n-k}],$$

where $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials of degree $k-1$, their coefficients depending only on a, b, and k.

Corollary Let F_n be the Fibonacci sequence. Then we have

$$(1) \sum_{a+b=n} F_aF_b = \frac{1}{5}[(n-1)F_n + 2nF_{n-1}], n \geq 1;$$

$$(2) \sum_{a+b+c=n} F_aF_bF_c = \frac{1}{50}[(5n^2-9n-2)F_{n-1} + (5n^2-3n-2)F_{n-2}], n \geq 2;$$

Theorem (Ohtsuka, 2012, The Fibonacci Quarterly)

$$(1)\ \left\lfloor \left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{F_k}\right)\right)^{-1} \right\rfloor = F_{n-2}$$

$$(2)\ \left\lfloor \left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{F_k^2}\right)\right)^{-1} \right\rfloor = \begin{cases} F_nF_{n-1} & \text{for any odd } n \geq 3; \\ F_nF_{n-1}-1, & \text{for any even } n \geq 3; \end{cases}$$

Theorem (Kuhapatanakul, 2013, Journal of Integer Sequences)

For Fibonacci numbers $\{F_k\}$,

$$(1)\ \left(\sum_{k=n}^{\infty}\frac{1}{\sum_{i=0}^{k}F_i}\right)^{-1} = F_n - 1$$

$$(2)\ \left(\sum_{k=n}^{mn}\frac{1}{F_k}\right)^{-1} = \begin{cases} F_{n-2}, & \text{if } m \geq 3,\ n \geq 2 \text{ is even}; \\ F_{n-2}-1, & \text{if } n \geq 3 \text{ is odd,} \end{cases}$$

Introduction

Periodic Fibonacci . . .

Related problem

Proof of the main theorem

4 Proof of the main theorem

Theorem (Wu Zhengang, Zhang Jin)

For any positive integer $n > m$, the mth-order linear recursive sequences $\{u_n\}$ is defined as follows

$$u_n = a_1u_{n-1} + a_2u_{n-2} + \cdots + a_{m-1}u_{n-m+1} + a_mu_{n-m},$$

with $a_1, a_2, \cdots, a_m \in \mathbb{N}$ and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 2$, and initial values $u_i \in \mathbb{N}$ for $0 \leq i < m$ and at least one of them is not zero. For any real number $\beta > 2$ and positive integer $1 \leq s < \lfloor \log_{\frac{\alpha}{a_1}} \alpha d \rfloor$, where $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ are the roots of the characteristic equation of u_n and $d^{-1} = \max\{|\alpha_1|, |\alpha_2|, \cdots, |\alpha_{m-1}|\}$, then there exists a positive integer n_2 such that

$$\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{a_1^{sk}}{u_k^s} \right)^{-1} - \left(\frac{u_n^s}{a_1^{sn}} - \frac{u_{n-1}^s}{a_1^{sn-s}} \right) \right\| = 0, (n \geq n_2).$$

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

4.1. Several Lemmas

- **Lemma 1.** Let $a_1, a_2, \cdots, a_m \in \mathbb{N}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $m \in \mathbb{N}$ with $m \geq 2$. Then for the polynomial

$$f(x) = x^m - a_1 x^{m-1} - a_2 x^{m-2} - \cdots - a_{m-1} x - a_m,$$

we have

(I). Polynomial $f(x)$ has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$.

(II). Other $m - 1$ zeros of $f(x)$ lie within the unit circle in the complex plane.

Proof of (I).

- For any positive integer $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $m \geq 2$,

$$f(a_1) = -a_2 a_1^{m-2} - \cdots a_{m-1} a_1 - a_m < 0,$$

$$f(a_1+1) > (a_1+1)^m - a_1((a_1+1)^{m-1} + (a_1+1)^{m-2} + \cdots + 1) > 0.$$

- Thus there exits a positive real zero α of $f(x)$ with $a_1 < \alpha < a_1 + 1$. According to Descarte's rule of signs, $f(x) = 0$ has at most one positive real root. So $f(x)$ has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$.

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

Proof of (II).

- Observe from (I) in Lemma 1, we have

$$\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \text{ then } f(x) > 0, \tag{3}$$

$$\text{if } x \in \mathbb{R} \text{ such that } 0 < x < \alpha, \text{ then } f(x) < 0. \tag{4}$$

- Let

$$g(x) = (x-1)f(x) = x^{m+1} - (a_1+1)x^m + (a_1 - a_2)x^{m-1} + (a_2 - a_3)x^{m-2} + \cdots + (a_{m-1} - a_m)x + a_m.$$

Since $f(x)$ has exactly one positive real zero α, $g(x)$ has two positive real zeros α and 1.

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

Observe that

$$\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \text{ then } g(x) > 0, \tag{5}$$

$$\text{if } x \in \mathbb{R} \text{ such that } 1 < x < \alpha, \text{ then } g(x) < 0. \tag{6}$$

To complete the proof of (II) in Lemma 1, it is sufficient to show that there is no zero on and outside of the unit circle.

- **Claim 1:** $f(x)$ has no complex zero z_1 with $|z_1| > \alpha$.
- **Claim 2:** $f(x)$ has no complex zero z_2 with $1 < |z_2| < \alpha$.
- **Claim 3:** On the circle $|z_3| = \alpha$ and $|z_3| = 1$, $f(x)$ has the unique zero α.

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

- **Lemma 2.** Let $m \geq 2$ and let $\{u_n\}_{n\geq 0}$ be an integer sequence satisfying the recurrence formula (1). Then the closed formula of u_n is given by

$$u_n = c\alpha^n + \mathcal{O}(d^{-n}) \qquad (n \to \infty)\,.$$

where $c > 0$, $d > 1$, and $a_1 < \alpha < a_1 + 1$ is the positive real zero of $f(x)$.

- **Proof** Let $\alpha, \alpha_1, \ldots, \alpha_t$ be the distinct roots of $f(x) = 0$, where $f(x) = 0$ is the characteristic equation of the recurrence formula. From Lemma 1 we know that α is the simple root of $f(x) = 0$, then let r_j for $j = 1, 2, \ldots, t$ denote the multiplicity of the root α_j. From the properties of mth-order linear recursive sequences,

$$u_n = c\alpha^n + \sum_{i=1}^{t} P_i(n)\alpha_i^n,$$

where

$$P_i(n) \in \mathbb{R}[n], \;\; \deg P_i(n) = r_i - 1, \;\; r_1 + r_2 + \cdots + r_t = m - 1, \;\; c \in \mathbb{R}.$$

- **Remark** For example, for positive integers $1 \le u, v, w \le t$, if α_u is the simple root of $f(x)$, then $P_u(n) = g_1$, where $g_1 \in \mathbb{R}$, and $\deg P_u(n) = 0$; if α_v is the double root of $f(x)$, then $P_v(n) = g_2 n + g_3$, where $g_2, g_3 \in \mathbb{R}$, and $\deg P_v(n) = 1$; if α_w is the multiple root of $f(x)$ with the multiplicity r_w, then $P_w(n) = b_1 n^{r_w - 1} + b_2 n^{r_w - 2} + \cdots + b_{r_w - 1} n + b_{r_w}$, where $b_1, b_2, \cdots, b_{r_w} \in \mathbb{R}$, and $\deg P_w(n) = r_w - 1$.

  From Lemma 1 we have $|\alpha_i| < 1$ for $1 \le i \le t$. Since each term of tail of above identity goes to 0 as $n \to \infty$, we can find constant $M \in \mathbb{R}$ and $d \in \mathbb{R}$ with $d > 1$ for $n > n_0$ such that

$$\left| \sum_{i=1}^{t} P_i(n) \alpha_i^n \right| \le \sum_{i=1}^{t} |P_i(n) \alpha_i^n| \le M d^{-n},$$

  which completes the proof (note that if all roots of $f(x)$ are distinct we can choose $d^{-1} = \max\{|\alpha_1|, |\alpha_2|, \cdots, |\alpha_{m-1}|\}$ and $M = m - 1$).

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

- **Lemma 3.** Let $m \geq 2$ and let $\{u_n\}_{n\geq 0}$ be an integer sequence satisfying the recurrence formula (1). Then for any positive integer s, we have

$$u_n^s = c^s\alpha^{sn} + O(\alpha^{sn-n}d^{-n}) \qquad (n \to \infty),$$

where $c > 0$, $d > 1$, and $a_1 < \alpha < a_1 + 1$ is the positive real zero of $f(x)$.

Proof of the Theorem

- From the geometric series as $q \to 0$, we have

$$\frac{1}{1 \pm q} = 1 \mp q + O(q^2) = 1 + O(q).$$

-

$$\begin{aligned}
\frac{a_1^{sk}}{u_k^s} = \frac{a_1^{sk}}{c^s \alpha^{sk} + O(\alpha^{sk-k} d^{-k})} &= \frac{a_1^{sk}}{c^s \alpha^{sk}(1 + O(\alpha^{-k} d^{-k}))} \\
&= \frac{a_1^{sk}}{c^s \alpha^{sk}} \left(1 + O(\alpha^{-k} d^{-k})\right) \\
&= \frac{a_1^{sk}}{c^s \alpha^{sk}} + O\left(\frac{a_1^{sk}}{\alpha^{sk+k} d^k}\right).
\end{aligned}$$

Introduction

Periodic Fibonacci . . .

Related problem

Proof of the main theorem

- Thus

$$
\begin{aligned}
\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{a_1^{sk}}{u_k^s} =& \frac{1}{c^s} \sum_{k=n}^{\lfloor \beta n \rfloor} \left(\frac{a_1}{\alpha}\right)^{sk} + O\left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{a_1^{sk}}{\alpha^{sk+k} d^k}\right) \\
=& \frac{\alpha^s}{\alpha^s - a_1^s} \cdot \left(\frac{a_1}{\alpha}\right)^{sn} - \frac{1}{\alpha^s - a_1^s} \cdot \left(\frac{a_1}{\alpha}\right)^{s\lfloor \beta n \rfloor} + O\left(\frac{a_1^{sn}}{\alpha^{sn+n} d^n}\right) \\
=& \frac{\alpha^s}{\alpha^s - a_1^s} \cdot \left(\frac{a_1}{\alpha}\right)^{sn} + O\left(\frac{a_1^{sn}}{\alpha^{sn}} \cdot \frac{a_1^{s\lfloor \beta n \rfloor - sn}}{\alpha^{s\lfloor \beta n \rfloor - sn}}\right) + O\left(\frac{a_1^{sn}}{\alpha^{sn}} \cdot \frac{1}{\alpha^n d^n}\right) \\
=& \frac{\alpha^s}{\alpha^s - a_1^s} \cdot \left(\frac{a_1}{\alpha}\right)^{sn} + O\left(\frac{a_1^{sn}}{\alpha^{sn}} \cdot h\right),
\end{aligned}
$$

where $h = \max\{\frac{a_1^{s\lfloor \beta n \rfloor - sn}}{\alpha^{s\lfloor \beta n \rfloor - sn}}, \frac{1}{\alpha^n d^n}\}$.

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

- Taking reciprocal, we get

$$\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\frac{a_1^{sk}}{u_k^s}\right)^{-1}=\frac{u_n^s}{a_1^{sn}}-\frac{u_{n-1}^s}{a_1^{sn-s}}+O\left(\frac{\alpha^{sn}}{a_1^{sn}}\cdot h\right).$$

- **Case 1.** If $h=\frac{a_1^{s\lfloor\beta n\rfloor-sn}}{\alpha^{s\lfloor\beta n\rfloor-sn}}$, for any real $\beta>2$ and positive integer s,

$$\frac{\alpha^{sn}}{a_1^{sn}}\cdot h=\frac{\alpha^{sn}}{a_1^{sn}}\cdot\frac{a_1^{s\lfloor\beta n\rfloor-sn}}{\alpha^{s\lfloor\beta n\rfloor-sn}}=\left(\frac{a_1}{\alpha}\right)^{s\lfloor\beta n\rfloor-2sn}<1.$$

Case 2. If $h=\frac{1}{\alpha^n d^n}$, for any positive $a_1\geq 2$, $1<\frac{\alpha}{a_1}<\alpha d$ holds, then for any positive integer s with

$$1\leq s<\lfloor\log_{\frac{\alpha}{a_1}}\alpha d\rfloor,$$

we have

$$\frac{\alpha^{sn}}{a_1^{sn}}\cdot h=\frac{\alpha^{sn-n}}{a_1^{sn}d^n}=\left(\frac{\alpha^{s-1}}{a_1^s d}\right)^n<1.$$

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

Related results

Theorem 2. Let $\{u_n\}$ be an mth-order sequence defined by (1) with the restriction $a_1, a_2, \cdots, a_m \in \mathbb{N}$ and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 2$. Let p and q be positive integers with $0 \leq q < p$. For any real number $\beta > 2$ and positive integer $1 \leq s < \lfloor \log_{\frac{\alpha}{a_1}} \alpha^p d^p \rfloor$, where $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ are the roots of the characteristic equation of u_n and $d^{-1} = \max\{|\alpha_1|, |\alpha_2|, \cdots, |\alpha_{m-1}|\}$, then there exist positive integers n_7, n_8 and n_9 depending on a_1, a_2, $\cdots$, and a_m such that

$$(a).\quad \left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-a_1)^{sk}}{u_k^s} \right)^{-1} - (-1)^{sn} \left(\frac{u_n^s}{a_1^{sn}} + \frac{u_{n-1}^s}{a_1^{sn-s}} \right) \right\| = 0, (n \geq n_7).$$

$$(b).\quad \left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{a_1^{spk+sq}}{u_{pk+q}^s} \right)^{-1} - \left(\frac{u_{pn+q}^s}{a_1^{spn+sq}} - \frac{u_{pn-p+q}^s}{a_1^{spn+sq-sp}} \right) \right\| = 0, (n \geq n_8).$$

$$(c).\quad \left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-a_1)^{spk+sq}}{u_{pk+q}^s} \right)^{-1} - (-1)^{spn+sq} \left(\frac{u_{pn+q}^s}{a_1^{spn+sq}} + \frac{u_{pn-p+q}^s}{a_1^{spn+sq-sp}} \right) \right\| = 0, (n \geq n_9).$$

Introduction
Periodic Fibonacci . . .
Related problem
Proof of the main theorem

Thank you!