Supergravity & Generalized Geometry

Possible reference: My book 'Geometry of String Theory Compactifications', CUP, 2022.

0. Introduction.

Supergravity: → Interesting geometrical problems / Approx. to string theory.

These lectures: esp. 'Type II', $D=10$

Often: * 'Vacuum' solutions $\mathrm{Mink}_d \times M_{10-d}$, $AdS_d \times M_{10-d}$ ($dS_d \times M_{10-d}$? One day...)

M_{10-d} compact (unless otherwise stated)

* Supersymmetry. Bosons := sections of TM, T^*M, (anti-) symm...

Fermions := sections of $\underset{\text{sp. bundle}}{SM} \otimes$ (bosons)

But: liberation from spinors is possible!

Prototype: Cov. constant spinor ⟷ Special Holonomy

1. Type II sugra. 1.1 Bosonic fields

$g_{\mu\nu}$ metric ; ϕ 'dilaton' function ; H 3-form 'NSNS' , F even/odd Form 'RR' IIA/IIB

* Bianchi id. $dH=0$ almost everywhere

more gen. $dH = \delta$ ← current loc. at 'NS5', Cod. 4

$H = dB$ ← locally. Really B is 'gerbe connection'

(Similar to $\underset{\text{curv.}}{f} = d\underset{\text{conn.}}{a}$ on $U(1)$ bd.)

(Similar to $U(1)$ fibr. with fiber deg. on Cod. 3)

'gauge' transf.: small $B \sim B + d\lambda$; large $B \sim B + \Lambda$ $\frac{1}{4\pi^2}\int \Lambda \in \mathbb{Z}$

* Flux quantization $\frac{1}{4\pi^2}\int H \in \mathbb{Z}$

Similar for RR:

$(d - H\wedge) F = 0$ a.e.

current: D-branes or O-planes dep. on 'sign'

$F = (d - H\wedge) C$ (subtlety in IIA for F_0) $(d-H\wedge)^2 = 0$

$C \begin{cases} \text{odd} & \text{IIA} \\ \text{even} & \text{IIB} \end{cases}$ again, connection.

gauge tr., flux quant. also ∃ for F.

* Self-duality: $F = *\lambda F$

$\lambda(\alpha_k) := (-1)^{\lfloor k/2 \rfloor} \alpha_k$

$*$ Action $\quad S = \frac{1}{(2\pi)^7} \int_{M_{10}} d^{10}x \sqrt{-g} \left(e^{-2\phi} \left(R_{10} + 4 |d\phi|^2 - \frac{1}{2} |H|^2 \right) - \frac{1}{4} \sum_k |F_k|^2 \right)$

<u>Exercise</u> $*$ Einstein eqs [long]

$\alpha_k \cdot \beta_k := \frac{1}{k!} \alpha_{\mu_1 \ldots \mu_k} \beta^{\mu_1 \ldots \mu_k}$

$|\alpha_k|^2 := \alpha_k \cdot \alpha_k$

<u>Rem</u> Heterotic: no F, but non-ab gauge field

$dH \neq 0$, dep. curvature

<u>1.2</u>

- Reminder <u>Fermions</u>

$*$ Eucl. (Mink) $\quad A = 1, \ldots, d$

$\{\gamma_a, \gamma_b\} = 2\delta_{ab} 1$ Cl. algebra $Cl(d)$ $\qquad \sim$ unique choice, dim $2^{\lfloor d/2 \rfloor}$

$\Rightarrow \frac{1}{2}\gamma_{ab}$: rep. of Lie alg. $so(d)$ $\qquad$ group: $\rho_S(\exp\lambda) = \exp(-\frac{1}{2}\lambda^{ab}\Gamma_{ab})$

$\gamma_{ab} = \gamma_{[a}\gamma_{b]}$ $\quad$ <u>spinor</u> rep. ρ_S $\qquad$ form group

act on v. sp. S $\qquad$ $Spin(d) \xrightarrow{2:1} O(d)$

$*$ <u>chirality</u> $\quad \gamma = c\gamma_1 \cdots \gamma_d \qquad \{\gamma, \gamma_a\} = 0 \qquad (\exists c \mid \gamma^2 = 1)$

(flat: $\gamma_1^2 = 1$, etc.)

d even: $\quad P_\pm = \frac{1}{2}(1 \pm \gamma)$ proj. on $\pm$ chirality

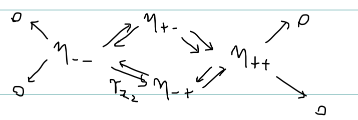
d odd $\quad \gamma \propto 1$

$*$ <u>Dist. basis</u>: $\quad d = 2 \qquad z = x^1 + ix^2 \qquad \{\gamma_z, \gamma_{\bar z}\} = 1 \qquad \gamma_z^2 = \gamma_{\bar z} = 0 \qquad S = \mathbb{C}^2$

$\gamma_{\bar z} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma_z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \eta_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad 0 \leftarrow \eta_- \underset{\gamma_z}{\overset{\gamma_{\bar z}}{\rightleftarrows}} \eta_+ \rightarrow 0$

$d = 4 \qquad z^1 = x^1 + ix^3 \qquad z^2 = x^2 + ix^4 \qquad \gamma_{z_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes 1_2 \qquad \gamma_{z_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad S = \mathbb{C}^2 \otimes \mathbb{C}^2$

$\eta_{--} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \gamma_{z_i} \eta_{--} = 0$



$\nearrow = \gamma_{z_1}$

$\searrow = \gamma_{z_2}$ etc.

<u>Def</u>. $\dim \mathrm{Ann}\,\eta = d/2$: η <u>pure</u>. $\quad \mathrm{Ann}\,\eta \rightsquigarrow$ choice of complex str.

<u>Fact</u>: all chiral η pure up to $d = 6$.

* $\gamma^{a_1..a_k} \equiv \gamma^{[a_1}.. \gamma^{a_k]}$ span End(S) (total dim $\sum \binom{k}{d} = 2^d$)

$\gamma^a \gamma^b = \frac{1}{2}\{\gamma^a,\gamma^b\} + \frac{1}{2}[\gamma^a,\gamma^b] = \delta^{ab}1 + \gamma^{ab}$

$\gamma^a \gamma^{b_1..b_k} = k\, \delta^{a[b_1}\gamma^{b_2..b_k]} + \gamma^{ab_1..b_k}$

$\overleftarrow{\gamma}_a C = \gamma_a C$ $\quad \overleftarrow{\gamma}_a C = C\gamma_a$

under 'Clifford map' $\quad dx^{a_1}\wedge..\wedge dx^{a_k} \overset{\checkmark}{\longleftrightarrow} \underset{(dx^{a_1}\wedge..\wedge dx^{a_k}\checkmark)}{\underset{\parallel}{\gamma^{a_1..a_k}}}$

$\overrightarrow{\gamma}^a \mapsto \iota_{\partial_b}\delta^{ab} + dx^a\wedge$

likewise, $\overleftarrow{\gamma}^a \mapsto (-\iota_{\partial_b}\delta^{ab} + dx^a\wedge)(-)^k$

$\Downarrow$

Rem $\Gamma^a := dx^a\wedge$, $\Gamma^{a+d} := \delta^{ab}\iota_{\partial_b}$

satisfy Clifford alg. $\{\Gamma^A,\Gamma^B\} = I^{AB}1$ $\quad I = \begin{pmatrix} 0 & 1_d \\ 1_d & 0\end{pmatrix}$ $\quad$ Cl(d,d) rel. to left, right Cl(d) action.
"Cl(d,d)"

Exercises * Check $[\gamma^{ab},\cdot] = 4\, dx^{[a}\wedge\iota^{b]}$
* what is $\overrightarrow{\gamma}$ on forms? $\quad$ ($\gamma = c\gamma^1...\gamma^d$ chir. matrix)

* curved: orth. basis e^a_μ ; ambig. $e^a_\mu \to \Lambda^a{}_b e^b_\mu$, $\Lambda \in O(d)$ $\qquad \eta \to \underline{\rho_S(\Lambda)}\eta$

$\gamma_\mu = e^a{}_\mu \gamma_a$

$\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}1$ $\qquad$ so that $\eta^\dagger\gamma_{\mu_1..\mu_k}\eta$ invariant

$w_1(M) = w_2(M) = 0 \quad \to \exists\ \Sigma$ with tr. fun $\rho_S(\Lambda_{\alpha\beta})$

$\exists$ cov. der $D_\mu\eta$. $\qquad \left[de^a + \omega^a{}_b\wedge e^b = 0,\ D_\mu = \partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\right]$

1.3

- Supersymmetry. $\exists$ Fermions: $\lambda^{1,2}$ sec SM $\quad \psi^{1,2}_M$ sec $SM\otimes T^*M$ $\quad$ chiral; 'Majorana' = real if we work with real Γ_μ

$\underset{\text{'supercharge'}}{\delta_\epsilon}(S_{bos} + S_{Ferm}) = 0$ $\qquad \delta\psi^1_\mu = \left(D_\mu - \frac{1}{4}\iota_\mu H\right)\epsilon^1 + \frac{e^\phi}{16}F\gamma_\mu\epsilon^2$ $\qquad D := \gamma^\mu D_\mu$ Dirac op.

Clifford map implicit: $\qquad \Gamma^\mu\delta\psi^1_\mu - \delta\lambda^1 = \left(D - \frac{1}{4}H - d\phi\right)\epsilon^1$ $\qquad$ & sim. for $\delta\psi^2_\mu$, $\delta\lambda$

$F \mapsto \not{F} \equiv \frac{1}{k!}F_{\mu_1..\mu_k}\gamma^{\mu_1..\mu_k}$

Rem $\delta\psi_M = D_M\epsilon$ reminiscent of $\delta g_{MN} = D_{(M}V_{N)}$

Rem $\delta(\text{bosons}) = 0 \Leftarrow$ ferm $= 0$ From now on.

Def Sol. susy if $\delta\psi_\mu = 0 = \delta\lambda$

Rem. $\phi = H = F = 0$ $\qquad$ Sol. susy $\leftrightarrow D_\mu\epsilon = 0$ $\qquad$ special holonomy

2. Vacua with $F=0$

2.1 $M_{10} = \mathrm{Mink}_4 \times M_6$

Factorization assumption:

$$\overset{4d}{}\qquad\overset{6d}{}$$
$$\epsilon^1 = \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_-$$
$$\epsilon^2 = \zeta_+ \otimes \eta^2_\mp + \zeta_- \otimes \eta^2_\pm$$

$\eta_- = \eta_+^*$ $\quad \zeta_+ = \zeta_-^*$

ζ_+ const. on Mink_4

Rem almost inevitable for min. # superch.
2 term for reality
Only one counterex., a funny one

Again $\phi = \text{const}$, $H = F = 0 \to$ susy 6d $\leftrightarrow D_\mu \epsilon = 0$

$m = 1, \dots, 6$ $\qquad \boxed{D_m \eta_+^{1,2} = 0}$

Holonomy group = Stab(η)

$\underline{\eta_+^2 = 0}$:

Recall: η_+ pure $\quad \gamma_{\bar\imath} \eta_+ = 0 \to \gamma_{\bar\imath \bar\jmath} \eta_+ = 0 \quad \gamma_{\imath\bar\jmath}\eta_+ = -\frac{1}{2}\delta_{\imath\bar\jmath}\eta_+$

$TM = T_{1,0}M \oplus T_{0,1}M$
almost cp. str. I | $I^2 = -1$ on TM
(ACS)

$\delta\eta_+ = -\frac{1}{2}\lambda^{ab}\gamma_{ab}\eta_+ = 0$

$\lambda^{\imath\jmath} = 0 \to \lambda^{\bar\imath\bar\jmath} = 0$

$\lambda^{\imath\bar\jmath}\delta_{\imath\bar\jmath} = 0$

$\to$ Lie alg. stab$(\eta) = su(3)$

$\to SU(3)$ holonomy, CY

$(\eta_-^\dagger \gamma_m \eta_+ = 0)$

$i J_{mn} := \eta_+^\dagger \gamma_{mn} \eta_+$

$\Omega_{mnp} := -\eta_-^\dagger \gamma_{mnp} \eta_+$

$\nabla_m \eta_+ \to \nabla J = \nabla \Omega = 0$

(a1) $-\frac{1}{6} J^3 = -\frac{i}{8}\Omega \wedge \bar\Omega$

(a2) $J \wedge \Omega = 0$

(d1) $dJ = 0$
(d2) $d\Omega = 0$

Rem. * $\Omega \to$ ACS I
* (d1) $\to I$ integrable
* (d1,2) $\to$ Kähler
* (a1) 'Monge-Ampère' $\to$ CY

Exercise
* Find susy eq. on $\eta_+^{1,2}$.

(a1,2) : SU(3) str. $\eta \leftrightarrow (J,\Omega)$ $\mathrm{Stab}(J,\Omega) = \mathrm{Stab}(\eta) = SU(3)$

Rem. $\eta^1, \eta^2 \neq 0$, not prop. $\rightarrow$ Hol $\subset SU(2)$ $\rightarrow$ $M_6 = T^2 \times K3$, T^6

2.2 $\eta^2 = 0$, $F = 0$; $\phi \neq$ const, $H \neq 0$

Again $\mathrm{Stab}(\eta^1_+) = SU(3)$; $\eta \leftrightarrow (J,\Omega) \mid (1),(2)$.

SU(3) structure.

SUSY $\Leftrightarrow$ $(D_m - \frac{1}{4} \iota_m H)\eta_+ = 0$, $(d\phi - \frac{1}{2}H)\eta_+ = 0$

$$\rightarrow\ i\nabla_m J_{np} = D_m \eta_+^\dagger \gamma_{np} \eta_+ + \eta_+^\dagger \gamma_{np} D_m \eta_+ = \frac{1}{8}\eta_+^\dagger [\gamma_{np}, \gamma^{rs}]\eta_+ H_{mrs} = i H_{mr[n} I_{p]}{}^r$$

$\Rightarrow d^c J = -H$ (d1')

$d^c := i(\bar\partial - \partial)$

$\Rightarrow 2i\partial\bar\partial J = -\delta$

Similar: $d(e^{-2\phi}\Omega) = 0 \Rightarrow \underline{I \text{ cpx.}}$ (d2') $K = \emptyset$

$d(e^{-2\phi} J^2) = 0$ (d3) 'conf. balanced'

Rem (a1,2), (d1',2',3)

Type II Analogue of Hull-Strominger system.

Exercise compute (d3')

2.3 $\eta^1, \eta^2 \neq 0$, $F = 0$; $\phi \neq$ const, $H \neq 0$

$\mathrm{Stab}(\eta^{1,2}_+) \longrightarrow$ two SU(3) structures

Alternative: $\eta_1 \otimes \eta_2^\dagger$ 'bispinor'; $\xleftarrow{\text{cl. map}}$ form $\Phi \mid \eta_1 \otimes \eta_2^\dagger = \not\Phi$

section of $\mathrm{End}(\Sigma)$

Rem $\eta_{1,2}$ pure $\rightarrow$ Φ pure as a $Cl(d,d)$ 'spinor'

Rem. $\Omega \mapsto I$ $\Phi \mapsto \mathcal{J}$, a.c.s on $T \oplus T^*$

$d\Omega = 0 \rightarrow I$ int. $d\Phi = 0 \rightarrow \mathcal{J}$ integrable: $L_{\mathcal{J}} = \{X \in T \oplus T^* \mid \mathcal{J}X = iX\}$ closed under Courant bracket.

Exercise Prove this using $[X,Y]_C := [X\cdot, \{d, Y\cdot\}] - (X \leftrightarrow Y)$

Lemma: η^1_+, η^2_+, metric can be rec. from $\Phi_\pm$, $\Phi_\pm = \eta^1_+ \otimes \eta^{2\dagger}_\pm$ Rem. $\Phi_\pm$ pure, satisfy 'compatibility' sim. to (1),(2).

Idea: $\mathrm{Ann}(\Phi_+) \cap \mathrm{Ann}(\Phi_-) \simeq \mathrm{Ann}(\eta^1)$ $\mathrm{Ann}(\Phi_+) \cap \mathrm{Ann}(\Phi_-^*) \simeq \mathrm{Ann}(\eta^2)$

$\cap$ $Cl(6,6)$, sec $T \oplus T^*$

Rem.

$\Gamma^a := dx^a \wedge$, $\Gamma^{a+6} := \delta^{ab}\iota_{\partial_b}$ $\qquad$ $g\mathrm{Stab}(\Phi) = SU(3,3)$ $\qquad$ $g\mathrm{Stab}(\Phi_1,\Phi_2) = SU(3)\times SU(3)$

$-\frac{1}{2}\Gamma_{AB}$: rep. of $o(d,d)$ Lie algebra $\qquad$ $\cap$ $o(6,6)$ gen. by $\overrightarrow{\gamma}_{mn}$, $\overleftarrow{\gamma}_{mn}$ $\qquad$ $\to SU(3)\times SU(3)$ str. on $T\oplus T^*$:= 'gen. SU(3) structure'

$A = 1,\dots,12$

Exercise $\eta^1 = \eta^2 \longrightarrow \Phi_+ = e^{-iJ}$ $\quad \Phi_- = \Omega$ $\quad$ single SU(3) str.

Use Fierz: $\underset{\mathrm{End}(S)}{\alpha} = \sum_{k=0}^{d} \frac{1}{2^{d/2}k!}\mathrm{Tr}(\alpha\gamma_{a_k\dots a_1})\gamma^{a_1\dots a_k}$

SUSY $\Leftrightarrow$ $\left(D_m - \frac{1}{4}\iota_m H\right)\eta^1_+ = 0$ $\quad$ $\left(D_m + \frac{1}{4}\iota_m H\right)\eta^2_+ = 0$ $\quad \Leftrightarrow \quad (d - H\wedge)(e^{-\phi}\Phi_\pm) = 0$

$\left(d\phi - \frac{1}{2}H\right)\eta^1_+ = 0$ $\quad$ $\left(d\phi + \frac{1}{2}H\right)\eta^2_+ = 0$ $\quad \Rightarrow$ gen. Kähler

For a quick proof:

$2d\alpha_\pm = \gamma^m \nabla_m \alpha_\pm \pm \nabla_m \alpha_\pm \gamma^m$, $\qquad H\wedge\alpha_\pm = H\wedge\alpha_\pm \pm \alpha_\pm \wedge H + \gamma^m \alpha_\pm \iota_m H \pm \iota_m H \alpha_\pm \gamma^m$ $\qquad$ Tomasiello (10.2.9)

omitting / symbol $\qquad$ Exercise Show these identities.

Rem. $(d - H\wedge)(e^{-\phi}\Phi_\pm) = 0 \Rightarrow$ gen. Kähler; but is stronger (similar to CY $\Rightarrow$ Kähler)

Rem. $dH = 0$: no compact sol. unless $\phi \sim \phi - \phi_0$ ('S-Fold')

alt. $dH = \delta$ ('O-planes' := 'negative D-branes')

3. Vacua with $F \neq 0$

3.1 $\mathrm{Mink}_4 \times M_6$ $\quad$ allow 'warped product': $ds^2_{10} = e^{2A} ds^2_{\mathrm{Mink}_4} + ds^2_6$ (fn. on M_6) $\qquad$ $\Phi_1 = \Phi_\pm$, $\Phi_2 = \Phi_\mp$ $\quad$ IIA / IIB

$\eta^1, \eta^2 \neq 0$, $F \neq 0$

$\phi \neq$ const, $H \neq 0$

SUSY $\to$ 6 equations on η^1, η^2; no apparent 'meaning' $\Longleftrightarrow$

$d_H(e^{2A-\phi}\Phi_1) = 0$ (ps1) $\qquad$ (f is F on M_6)

$d_H(e^{A-\phi}\mathrm{Re}\,\Phi_2) = 0$ (ps2)

$d_H(e^{3A-\phi}\mathrm{Im}\,\Phi_2) = e^{4A} * \lambda f$ (ps3)

Rem (ps3) can be replaced with $f = \underbrace{\mathcal{J}_\pm d_H}_{\text{gen. of } d^c = i(\bar\partial - \partial)}(e^{-A-\phi}\mathrm{Im}\,\Phi_2)$ $\quad \to d_H \mathcal{J}_\pm d_H(e^{-A-\phi}\mathrm{Im}\,\Phi_2) = \delta$

Rem if M_6 noncompact, slight generalization is possible. If M_6 is compact, O-planes still necessary.

Rem Φ are 'gen. calibrations' (lead to min. 'energy' rather than volume)

Ex. $\Phi_1=\Omega$, $\Phi_2=ie^{-iJ}$ (psp 1-3): $d(e^{3A-\phi}\Omega)=0$ $d(e^{2A-\phi}J)=0$ → conf. Kähler

∃ many sol ! $dd^c(e^{-A}(1-\frac{J^2}{2}))=\delta$ supp(δ): points, divisors

Ex. $\Phi_1=\Omega$ $\Phi_2=ie^{-iJ}$ (psp 1-3): sim. to HS system ('S-dual')

Ex. $\Phi_1=e^{-iJ}$ $\Phi_2=\Omega$ (psp 1-3): $dJ=0$ $d(e^{-A}\mathrm{Re}\,\Omega)=0$

$\underbrace{dd^\wedge}_{\text{'symp. }\partial\bar\partial\text{'}}(e^{-A}\mathrm{Im}\,\Omega)=\delta$ supp(δ) = 'slag'

symplectic non-Kähler: 'mirror' of HS

→ rel. to G_2 holonomy

Rem. String corrections → Evidence for compact solutions even without O-planes on complex or symp. non Kähler.

3.2 $AdS_4\times M_6$

$\eta^1,\eta^2\neq 0$, $F\neq 0$

$\phi\neq$ const, $H\neq 0$

$A\neq 0$

susy → 6 equations on η^1,η^2; no apparent 'meaning'

$\Longleftrightarrow$

$$d_H(e^{2A-\phi}\Phi_1)=2\mu e^{A-\phi}\mathrm{Re}\,\Phi_2$$

$$d_H(e^{3A-\phi}\mathrm{Im}\,\Phi_2)=3\mu e^{2A-\phi}\mathrm{Im}\,\Phi_1\mp e^{4A}*\lambda f$$

$\mu=\sqrt{-\Lambda_3}$

IIA

IIB

Ex IF $\Phi_1=e^{-iJ}$ $\Phi_2=\Omega$ → $dJ=2m\,\mathrm{Re}\,\Omega$ → 'half-flat'

$d\Lambda d\,\mathrm{Im}\,\Omega=10m^2\,\mathrm{Re}\,\Omega$

Ex. * nearly-Kähler,

* $Tw(qk)$

* (top. S^2) ↪ M_6 → KE_4

(similar to Calabi's Ansatz)

'Extended' susy: multiple sol. $\Phi_{1,2}$ with same fields $g_{\mu\nu},\phi,H,F$

Rem Fashionable question:

can make $\mathrm{diam}(M_6)\ll\frac{1}{\mu}$?

Same question for SE, weak G_2, ...

4. Other dimensions $\left.\begin{matrix}\mathrm{Mink}_d\\ \mathrm{AdS}_d\end{matrix}\right\}\times M_{10-d}$

4.1 $d>4$: part case of previous discussion

Ex $\mathrm{Mink}_4\times M_6$: $M_6=\mathbb{R}^2\times M_4 \longrightarrow \mathrm{Mink}_6\times M_4$

$\Phi_1=e^A(dx^4+i\,dx^5)\wedge\phi_1$

$\Phi_2=(1+ie^{2A}dx^4\wedge dx^5)\wedge\phi_2 \longrightarrow$

$\psi_\alpha=$

$d_H(e^{2A-\phi}\mathrm{Re}\phi_2)=0$

$d_H\left(e^{4A-\phi}\begin{pmatrix}\mathrm{Re}\phi_1\\ \mathrm{Im}\phi_1\\ \mathrm{Im}\phi_2\end{pmatrix}\right)=0$

$f=\mp e^{-\phi}*\lambda(dA\wedge\mathrm{Re}\phi_2)$

Ex. $\mathrm{Mink}_6\times M_4$: $M_4=\mathbb{R}_+\times M_3 \quad\longrightarrow \mathrm{AdS}_7\times M_3$

$ds^2_{M_4}=\frac{dr^2}{r^2}+ds^2_{M_3}$

$e^{2A_6}=r^2e^{2A_7}$

$\phi_1\to$ Id. structure on TM $\longrightarrow$ Full classification!

$\alpha=\alpha(z)$ piecewise cubic, continuous

$$ds^2_{M_3}=\sqrt{-\frac{\alpha''}{\alpha}}\left(dz^2+\frac{\alpha^2}{\alpha'^2-2\alpha\alpha''}ds^2_{S^2}\right)$$

Ex. Similar strategy: $\mathrm{AdS}_5\times M_5$. * Most famous sol. $M_5=SE_5$.

* $\exists$ also 'gen. SE'. But all known sol. are cont. deformations of SE except one ('Pilch–Warner')

Exercise $CY=\mathrm{cone}(SE)$

Find characterization of SE in terms of forms.

4.2 $d<4$: new 'pairing eqs.' appear

Ex $d=3$: $\left.\begin{matrix}\mathrm{Mink}_3\\ \mathrm{AdS}_3\end{matrix}\right\}\times M_7$:

$d_H(e^{A-\phi}\psi_2)=cF$

$d_H(e^{2A-\phi}\psi_1)+2\mu e^{A-\phi}\psi_2=e^A*\lambda f$

$(\psi_2,f)=4\mu e^{-\phi}(1-c^2e^{-4A})$

$\shortparallel$

$\frac{(\psi_2\wedge\lambda(f))_7}{\mathrm{vol}_7}$ 'Chevalley–Mukai pairing'

ψ_1,ψ_2: gen. G_2 structure

$G_2\times G_2$ str. on $T\oplus T^*$

Exercise $\psi=\eta_1\eta_2^t$ for $\eta^1=\eta^2$ Majorana ($=$ real) (γ_m purely im.)

use: Fierz for odd dim

Exercise

ψ in terms of $\Phi_{1,2}$

Rem $M_7=\mathbb{R}\times M_6 \rightsquigarrow \Phi_{1,2}$ gen. $SU(3)$ str. on M_6, evolution eqs.

(similar to Hitchin's $G_2\rightsquigarrow$ half-flat on M_6)

5. Exc. geometry. $\quad$ $\mathrm{Mink}_4 \times M_6$, to fix ideas.

Extend further $T \oplus T^* \rightsquigarrow T \oplus T^* \oplus \Lambda^{\pm} T^* \oplus \Lambda^5 T^* \oplus (\Lambda^6 T^* \otimes T^*) \equiv E$

str. group $SO(6,6) \subset E_7$!

(Rough)
Origin of the summands:

- $T \oplus T^*$: diff, gauge tr. for NSNS $(\delta B = d\lambda)$
- $\Lambda^{\pm} T^*$: gauge tr. for RR $\delta C = d_H \hat{\lambda}$ $\qquad$ $\hat{\lambda}$ even IIA, odd IIB
- $\Lambda^5 T^*$: ——— dual NSNS, $*H = d\tilde{B}$ $\quad$ $\delta \tilde{B} = d\lambda_5$
- $(\Lambda^6 T^* \otimes T^*)$ 'dual diffeomorphisms'

$\eta^{1,2} \rightsquigarrow \Phi^{1,2}$, reduce $SO(6,6) \longrightarrow SU(3) \times SU(3)$

'ψ' — $E_7 \longrightarrow SU(7)$

Diff. conditions:

susy $\Leftrightarrow$ $SU(7)$ holonomy on E

Rem. For $AdS_4 \times M_6$, 'singlet intrinsic torsion'

Similar statements in other dimensions.