Introduction to generalized Kähler geometry

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Two lectures delivered at the 2024 TIMS Summer Program in Geometry, on July 15 and 16, at National Taiwan University. An introduction to generalized Kähler geometry, covering both the bi-Hermitian and generalized geometry approaches.

1 Introduction

Generalized Kähler geometry was discovered by physicists Gates, Hull, and Roček in 1984 [15], when they sought to generalize the earlier work of Zumino, who showed that a Kähler structure on the target of the 2-dimensional sigma model endows the model with an action by the $N = (2, 2)$ supersymmetry algebra. They observed that the same occurs if the target is endowed with a generalized Kähler structure.

The literature on generalized Kähler geometry may be very roughly summarized as follows:

1. The (very large) physics literature, of which some key examples are [15, 30, 28], and most recently [27].
2. Relevant literature from complex geometry, especially [2].
3. The link to Hitchin's generalized geometry [25, 19, 22, 23]
4. The construction of many examples [26, 21], for example by generalized Kähler reduction [7, 8].
5. Hodge theory for generalized Kähler structures [20, 9, 3]
6. T-duality and generalized geometry [1]
7. Deformation theory of Generalized Kähler metrics [16, 24],
8. Curvature of generalized Kähler manifolds, and Kobayashi-Hitchin correspondence for vector bundles over generalized Kähler manifolds [18, 17]
9. Generalized Ricci Flow [14], which includes generalized Kähler–Ricci flow, a variant of pluriclosed flow.
10. Generalized Kähler geometry and symplectic groupoids, the generalized Kähler potential [5, 38]

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2 Generalized geometry

In generalized geometry, instead of using the tangent bundle TM of the manifold M to model geometric structures, we use an extension of this bundle. The only example we shall study in this course is

$$\mathbb{T}M = TM \oplus T^*M,$$

which is useful for understanding T-duality, Mirror symmetry, and Type I and II string theories. There is also

$$TM \oplus \mathfrak{g}_M \oplus T^*M,$$

where $\mathfrak{g}_M$ is the adjoint bundle of a principal G-bundle, which is useful for understanding the Hull–Strominger system and Heterotic string theory. See these references: [4, 13, 33]. There are even more complicated examples, useful for capturing other supergravity theories, see for example [6] and its references. All of the above are examples of *Courant algebroids*, introduced in [29]. Very generally, the Courant algebroid is supposed to be part of the background or substrate, on top of which the geometry is defined. Once we have introduced the simplest kind of Courant algebroid, we will explore the concept of a *generalized Riemannian metric* on it, touching on its associated Hodge decomposition.

2.1 The Courant algebroid $\mathbb{T}M$

Split signature metric and spinors

A section $X + \xi \in C^\infty(\mathbb{T}M)$ acts on a differential form $\rho \in \Omega(M)$ via interior and exterior product:

$$(X + \xi) \cdot \rho = i_X\rho + \xi \wedge \rho.$$

If we square this action, we obtain

$$(X + \xi) \cdot ((X + \xi) \cdot \rho) = i_X(\xi \wedge \rho) + \xi \wedge i_X\rho = (i_X(\xi))\rho,$$

so that if we define a symmetric bilinear form on the bundle $\mathbb{T}M$ as follows:

$$\langle X + \xi, Y + \eta\rangle = \tfrac{1}{2}(i_X\eta + i_Y\xi),$$

we obtain a metric of signature (n, n) for $n = \dim M$ on $\mathbb{T}M$ such that the squared action satisfies

$$(X + \xi) \cdot ((X + \xi) \cdot \rho) = \langle X + \xi, X + \xi\rangle\rho.$$

This implies that the Clifford algebra bundle $Cl(\mathbb{T}M, \langle\cdot,\cdot\rangle)$ acts on the bundle $\wedge^\bullet T^*M$ of differential forms. This representation is actually the spin representation of the real Clifford algebra of signature (n, n). This representation is irreducible for the action of the Clifford algebra, but if we consider the Spin subgroup $Spin(n, n)$, then this decomposes into a sum of irreducibles: the *even* and *odd* spinors, corresponding to differential forms of even and odd degree, respectively:

$$S = \wedge^{\bullet} T^* M = \wedge^{ev} T^* \oplus \wedge^{od} T^* = S^+ \oplus S^-.$$

In conclusion, on any manifold M, the natural bundle $\mathbb{T}M$ is endowed with a metric of split signature, and we may view the differential forms of M as its spinors.

Exercise 2.1. Prove that the bundle of Lie algebras $\mathfrak{so}(\mathbb{T}M, \langle \cdot, \cdot \rangle)$ is naturally isomorphic to

$$\wedge^2 TM \oplus \mathsf{End}(TM) \oplus \wedge^2 T^* M.$$

Show that any section $\beta + A + B$ of the above defines the following block endomorphism of $\mathbb{T}M$:

$$\begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix},$$

where $B \in \wedge^2 T^*$, for example, determines the transformation $B : X + \xi \mapsto i_X B$. Use this to compute the Lie bracket. □

Exercise 2.2. The Lie algebra $\mathfrak{so}(\mathbb{T}M, \langle \cdot, \cdot \rangle)$ acts on spinors, and in particular, $B \in \Omega^2(M)$ acts on $\rho \in \Omega^{\bullet}(M)$ via $\rho \mapsto -B \wedge \rho$. This action of $\Omega^2(M)$ is called a *B-field transformation* in physics. Prove that the actions intertwine, i.e. show that

$$-B \wedge ((X + \xi) \cdot \rho) = (B(X + \xi)) \cdot \rho + (X + \xi) \cdot (-B \wedge \rho).$$

By exponentiating this action, show that

$$e^{-B}((X + \xi) \cdot \alpha) = (e^B(X + \xi)) \cdot e^{-B} \alpha.$$

In other words, if $c(X + \xi) = (X + \xi) \cdot$ is the operator of Clifford action by $X + \xi$, then the above equation may be interpreted as follows:

$$c(e^B(X + \xi)) = e^{-B} \circ c(X + \xi) \circ e^B. \qquad (1)$$

□

The spin representation has a natural bilinear form called the Chevalley pairing: for $\alpha, \beta \in \Omega^{\bullet}(M)$, their pairing is the top degree form

$$\langle \alpha, \beta \rangle_S = (\alpha \wedge \beta^{\top})_{\mathrm{top}},$$

where $(\rho)_{\mathrm{top}}$ denotes the component of degree $\dim M$ of ρ, and $\beta \mapsto \beta^{\top}$ is the reversal anti-automorphism of the differential forms, i.e.

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k})^{\top} = dx^{i_k} \wedge \cdots \wedge dx^{i_1}.$$

In other words, $\beta^{\top} = (-1)^{k(k-1)/2} \beta$ for β of degree k.

Exercise 2.3. Show the identity, for all $\alpha, \beta \in \Omega^\bullet(M)$ and $X + \xi \in \mathbb{T}M$,

$$\langle (X+\xi)\cdot\alpha, (X+\xi)\cdot\beta\rangle_S = \langle X+\xi, X+\xi\rangle\langle\alpha,\beta\rangle_S,$$

and conclude that the Chevalley pairing is invariant under the action of the identity component of $Spin(\mathbb{T}M, \langle\cdot,\cdot\rangle)$. In particular, check explicitly that for $B \in C^\infty(\wedge^2 T^*M)$,

$$\langle e^B\alpha, e^B\beta\rangle_S = \langle\alpha,\beta\rangle_S.$$

□

Exercise 2.4. Show the identity

$$\langle\alpha,\beta\rangle_S = (-1)^{n(n-1)/2}\langle\beta,\alpha\rangle_S. \tag{2}$$

Write the Chevalley pairing explicitly in the case of a 4-dimensional, 3-dimensional, and 2-dimensional manifold. Verify that the Chevalley pairing is symmetric in the first case, and skew-symmetric in the second and third cases. □

The Courant bracket

The Lie bracket of vector fields is dual to the de Rham exterior derivative, in a sense made precise by the following identity, for all vector fields X, Y and differential forms ρ:

$$i_{[X,Y]}\rho = [[d, i_X], i_Y]\rho.$$

In view of our earlier discussion of the action of $\mathbb{T}M$ on forms, we may extend the Lie bracket to a *Courant* bracket, as follows. Recall that $c(X+\xi) = (X+\xi)\cdot$ denotes the Clifford action of a section of $\mathbb{T}M$.

$$c([X+\xi, Y+\eta]) = [[d, c(X+\xi)], c(Y+\eta)]. \tag{3}$$

Exercise 2.5. With the above definition, prove that

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - i_Y d\xi.$$

□

Note that this bracket is not skew-symmetric, and satisfies

$$[X+\xi, X+\xi] = d\langle X+\xi, X+\xi\rangle,$$

or, more generally,

$$[X+\xi, Y+\eta] + [Y+\eta, X+\xi] = 2d\langle X+\xi, Y+\eta\rangle.$$

In particular, if sections are taken from an isotropic subbundle of $\mathbb{T}M$, the bracket will be skew-symmetric.

In fact, the Courant bracket satisfies a version of the Jacobi identity, as follows.

Exercise 2.6. Using the definition (3), prove the Jacobi identity for the Courant bracket, i.e.

$$[[X+\xi, Y+\eta], Z+\zeta] = [X+\xi, [Y+\eta, Z+\zeta]] - [Y+\eta, [X+\xi, Z+\zeta]].$$

□

Remark 2.7. The Courant bracket is close to being a Lie algebra. In fact, as shown in [32], it defines a L_∞ algebra structure on the following complex, concentrated in degrees -1, 0:

$$C^\infty(M, \mathbb{R}) \xrightarrow{d} C^\infty(\mathbb{T}M) .$$

The binary bracket (of degree zero) vanishes in degree -1 and is the skew-symmetrization of the Courant bracket in degree 0. Between degrees -1 and 0, the bracket is $[X+\xi, f] = \frac{1}{2}X(f)$, and finally the ternary bracket (of degree -1) has only one component, namely

$$[X+\xi, Y+\eta, Z+\zeta] = \tfrac{1}{3}(\langle [X+\xi, Y+\eta], Z+\zeta\rangle + c.p.).$$

□

It is natural to ask whether any part of the Lie algebra $\mathfrak{so}(\mathbb{T}M, \langle\cdot,\cdot\rangle)$ acts in such a way as to preserve the Courant bracket. Focusing on B-field transformations, we obtain the following identity: let $U = X+\xi$, $V = Y+\eta$ and $B \in \Omega^2(M)$. Then from the following identity:

$$[[d, e^{-B}c(U)e^{B}], e^{-B}c(V)e^{B}] = e^{-B}[[e^{B}de^{-B}, c(U)], c(V)]e^{B},$$

and using the fact that

$$e^{B}de^{-B} = d - dB \wedge \cdot,$$

we conclude that under the condition that B is closed, the B-field transformation e^{B} is a symmetry of the Courant bracket, namely

$$[e^{B}(X+\xi), e^{B}(Y+\eta)] = e^{B}[X+\xi, Y+\eta].$$

Exercise 2.8. Fill in the details in the above argument. Also, try to prove the fact that closed B-field transformations are the only sections of $\mathfrak{so}(\mathbb{T}M, \langle\cdot,\cdot\rangle)$ that preserve the Courant bracket. See [19] for a proof.

□

In the above argument, we saw that if $B \in \Omega^2(M)$ is *not closed*, then it does not preserve the Courant bracket; instead it takes the Courant bracket to a *twisted* Courant bracket, that is,

$$[e^{B}(U), e^{B}(V)] = e^{B}[U, V]_{dB},$$

where the bracket on the right hand side is defined in the same way as the Courant bracket, but for the differential

$$(d - dB \wedge \cdot) : \Omega^{ev/od}(M) \to \Omega^{od/ev}(M).$$

Generalizing the above operator slightly, we may define, for any closed 3-form H, a *twisted de Rham* operator

$$d_H = (d - H \wedge \cdot) : \Omega^{ev/od}(M) \to \Omega^{od/ev}(M),$$

and a *twisted* Courant bracket

$$c([X + \xi, Y + \eta]_H) = [[d_H, c(X + \xi)]c(Y + \eta)].$$

Exercise 2.9. Prove that the twisted Courant bracket is given by

$$[X + \xi, Y + \eta]_H = [X, Y] + L_X\eta - i_Y d\xi + i_Y i_X H.$$

□

In the study of 2-dimensional sigma models in physics, the closed 3-form H is known as the Wess–Zumino term, and in string theory it is known as the Neveu–Schwarz 3-form flux. In these theories, it is important that H has integral periods, and in fact it should be viewed as the curvature of a $U(1)$ gerbe with connection and curving.

Given a manifold M equipped with a closed 3-form H, the tuple $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$ is known as an *exact Courant algebroid*. The action of B-field transformations provides isomorphisms

$$[e^B(U), e^B(V)]_H = e^B[U, V]_{H+dB} \tag{4}$$

between Courant algebroids whose 3-forms are cohomologous. Indeed, as shown in [35], exact Courant algebroids are classified by the cohomology class $[H] \in H^3(M, \mathbb{R})$, known as the Ševera class.

2.2 Generalized metrics

The structure group of the metric bundle $\mathbb{T}M$ is $O(n, n)$; a reduction to the maximal compact subgroup $O(n) \times O(n)$ is called a *generalized Riemannian metric.*

Definition 2.10. A *generalized Riemannian metric* is a maximal positive-definite subbundle

$$V_+ \subset \mathbb{T}M. \tag{5}$$

The orthogonal complement $V_- = V_+^\perp$ relative to the split-signature metric is then maximally negative-definite, and we have a decomposition, orthogonal with respect to $\langle\cdot,\cdot\rangle$, as follows:

$$\mathbb{T}M = V_+ \oplus V_-.$$

Since the subbundles TM, T^*M of $\mathbb{T}M$ are null, in fact maximal isotropic, it follows that the projection along T^*M, i.e.

$$\pi : \mathbb{T}M \to TM,$$

defines an isomorphism of bundles

$$V_\pm \xrightarrow[\pi]{\cong} TM \ .$$

Exercise 2.11. Prove that a generalized Riemannian metric is given by the graph of a general 2-tensor $g + b \in C^\infty(S^2T^* \oplus \wedge^2 T^*)$ whose symmetric part g is positive-definite. That is,

$$V_+ = \{X + g(X) + b(X) \ : \ X \in TM\}.$$

Conclude that the orthogonal complement is then given by

$$V_- = \{X - g(X) + b(X) \ : \ X \in TM\}.$$

□

Since a generalized Riemannian metric determines the decomposition (5), it can be described in terms of an operator $\mathbb{G} : \mathbb{T}M \to \mathbb{T}M$ as follows:

$$\mathbb{G} = 1|_{V_+} + (-1)|_{V_-}.$$

Exercise 2.12. If the generalized Riemannian metric is given by $g+b$ as above, show that the corresponding operator is, in block form,

$$\mathbb{G} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}.$$

□

Generalized Hodge star

Assume we are on an oriented manifold, and choose an oriented orthonormal basis $(e_1, \ldots, e_n)$ for V_+ over any point. Then the product

$$* = e_n \cdots e_1 \in Cl(V_+) \subset Cl(\mathbb{T}M)$$

is a well-defined element of the Clifford algebra, independent of the chosen basis, and is therefore a global section of the Clifford algebra bundle. We call this element the *generalized Hodge star*. It acts on differential forms via the spin representation.

Exercise 2.13. Prove that the classical Hodge star $\star$ can be obtained from $*$ as follows. Assume $b = 0$, so that $V_+ = \mathrm{Gr}(g)$, for g a Riemannian metric. Then the Hodge star of g is

$$\star\rho = (*\rho)^\top.$$

Hint: Choose an oriented orthonormal basis $(v_1, \ldots, v_n)$ for TM, with dual basis $(v^i = g(v_i))_{i=1}^n$, so that $(e_i = v_i + v^i)_{i=1}^n$ defines an oriented orthonormal basis for V_+. Then verify using the Clifford action that

$$(v_n + v^n)\cdots(v_1 + v^1)\cdot(v^1 \wedge \cdots \wedge v^k) = v^n \wedge \cdots \wedge v^{k+1}.$$

$\square$

Exercise 2.14. Prove the identity

$$*^2 = (-1)^{n(n-1)/2}. \tag{6}$$

$\square$

Combining the above identity (6) with the identity (2) for the Chevalley pairing, we see that the top degree differential form

$$\langle \alpha, *\beta \rangle_S \tag{7}$$

is symmetric in α and β. In fact, when $b = 0$, we have the identity

$$\langle \alpha, *\beta \rangle_S = (\alpha \wedge (*\beta)^\top)_n = (\alpha \wedge \star\beta)_n = g(\alpha, \beta)\mathrm{vol}_g,$$

where $g(\alpha, \beta)$ is the induced Riemannian metric on differential forms and $\mathrm{vol}_g = \star 1 = \det(g)^{1/2}$ is the Riemannian volume form.

Exercise 2.15. Let $*_{g+b}$ be the generalized Hodge star for the generalized metric $g + b$. Using (1), prove the identity

$$*_{g+b} = e^{-b} *_g e^b.$$

Conclude that $\mathrm{vol}_{g+b} = \langle 1, *_{g+b} 1 \rangle$ defines a volume form, called the Born-Infeld or generalized Riemannian volume form, and verify that

$$\mathrm{vol}_{g+b} = (1 + |b|_g^2 + |\tfrac{b\wedge b}{2!}|_g^2 + |\tfrac{b\wedge b\wedge b}{3!}|_g^2 + \cdots)\mathrm{vol}_g. \tag{8}$$

$\square$

Let $(v_i)_{i=1}^n$ be an oriented orthonormal basis with respect to g. As a result, $(v_i + (g + b)(v_i))_{i=1}^n$ is an oriented orthonormal basis for V_+. Computing the corresponding Born-Infeld volume, we obtain

$$\begin{aligned} \langle 1, *_{g+b} 1 \rangle &= (g + b)(v_1) \wedge \cdots \wedge (g + b)(v_n) && (9) \\ &= \det(g + b)(v_1 \wedge \cdots \wedge v_n) && (10) \\ &= \det(g + b)\det(g)^{-1/2}. && (11) \end{aligned}$$

Combining this with (8), we conclude that

$$\det(g + b) = |e^b|_g^2 \det g.$$

Generalized harmonic forms

With these preliminaries out of the way, we may use (7) to define an L^2 inner product on spinors:

Definition 2.16. The Born-Infeld inner product of $\alpha, \beta \in S$ is defined to be

$$h(\alpha, \beta) = \int_M \langle \alpha, *\beta \rangle_S.$$

This is the appropriate L^2 metric for developing the Hodge theory of the differential d_H.

Exercise 2.17. Show that on a compact manifold M,

$$\int_M \langle d_H \alpha, \beta \rangle_S = (-1)^{\dim M} \int_M \langle \alpha, d_H \beta \rangle_S.$$

Conclude that the adjoint of d_H with respect to the Born-Infeld inner product is as follows:

$$d_H^* = (-1)^{\dim M} *^{-1} d_H * . \tag{12}$$

□

Since the operators d_H, d_H^* differ from d, d^* by lower-order terms, it follows that the associated Dirac operator $d_H + d_H^*$ is elliptic, just as in the case of $b = 0$. The associated *twisted Laplacian*

$$\Delta_H = (d_H + d_H^*)^2 = d_H d_H^* + d_H^* d_H$$

is then elliptic as an operator on forms of fixed parity. As usual, we obtain a Hodge decomposition theorem:

Theorem 2.18. *On a compact generalized Riemannian manifold, any even or odd d_H cohomology class has a unique Δ_H-harmonic representative.*

Exercise 2.19. Write the twisted Laplacian explicitly, in terms of the usual Laplacian plus lower-order terms. Also, demonstrate that the B-field transformation e^B defines an isomorphism from Δ_H-harmonic forms to Δ_{H+dB}-harmonic forms. □

3 Generalized Kähler geometry

A Riemannian manifold (M, g) is Kähler when there is an integrable complex structure I, compatible with g in the sense that $I \in \mathfrak{so}(TM, g)$ is an infinitesimal symmetry, i.e.

$$gI + I^* g = 0,$$

and such that the associated Hermitian form $\omega = gI$ is *closed*, thereby defining a symplectic form. The integrability of I and the closure of ω may be subsumed in the equivalent condition that the Levi-Civita connection preserves I, i.e. $\nabla I = 0$, giving an alternative definition of Kähler structure as a Riemannian manifold with holonomy (of the Levi-Civita connection) contained in $U(n)$. We now develop the notion of generalized Kähler structure, which makes use of a generalized complex structure compatible with a generalized Riemannian metric. To describe the integrability of such a generalized complex structure, we need the notion of a Dirac structure, which is a generalization of an integrable distribution, or foliation.

3.1 Dirac structures

Fix a manifold M with closed 3-form H, and let $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$ be the associated Courant algebroid.

Definition 3.1. A Dirac structure is an involutive maximal isotropic subbundle $L \subset \mathbb{T}M$.

Exercise 3.2. Prove that a maximal isotropic subbundle $L \subset \mathbb{T}M$, if it is transverse to $T^*M \subset \mathbb{T}M$, must be the graph of a 2-form, i.e. we have

$$L = \{X + \omega(X) \ : \ X \in TM\},$$

for some $\omega \in \Omega^2(M)$. Then show that $C^\infty(L)$ is closed under the Courant bracket if and only if $d\omega = H$. In particular, in the case $H = 0$, closed 2-forms correspond to Dirac structures transverse to T^*M. □

Exercise 3.3. Prove that a maximal isotropic subbundle $L \subset \mathbb{T}M$, if it is transverse to $TM \subset \mathbb{T}M$, must be the graph of a bivector field, i.e. we have

$$L = \{\beta(\xi) + \xi \ : \ \xi \in T^*M\},$$

for some bivector field $\beta \in C^\infty(\wedge^2 TM)$. Then show $C^\infty(L)$ is closed under the Courant bracket if and only if

$$[\beta, \beta] = (\wedge^3\beta)(H),$$

where on the left we have the Schouten bracket of multivector fields. Such a bivector field is called a *twisted Poisson structure* [36], and when $H = 0$ this is simply a Poisson structure. □

In general, a Dirac structure need not be transverse to TM or to T^*M, and its projection to TM defines a distribution with potentially nonconstant rank. In a neighbourhood U of a point where $\pi_T(L)$

has constant rank, we have a distribution $\iota : F \hookrightarrow TU$ and a 2-form $\omega \in C^\infty(U, \wedge^2 F^*)$, such that

$$L|_U = \{X + \xi \in \mathbb{T}U \ : \ X \in F,\ \iota^*\xi = i_X\omega\}.$$

Involutivity of L is then equivalent, in U, to the condition that F is integrable to a foliation and that $d\omega = \iota^* H$. In this way, we see that Dirac structures are generalizations of Poisson structures in which we have a singular foliation whose leaves are equipped with (possibly degenerate) 2-forms which serve as primitives for the pullback of H.

3.2 Generalized complex structures

Recall that a complex structure is an endomorphism $I \in C^\infty(\mathsf{End}(TM))$ such that $I^2 = -1$ and such that the $+i$-eigenbundle $T_{1,0}M \subset TM \otimes \mathbb{C}$ is involutive, meaning

$$[T_{1,0}M, T_{1,0}M] \subset T_{1,0}M.$$

We now generalize this to the exact Courant algebroid over M defined by the closed 3-form $H \in \Omega^3(M)$.

Definition 3.4. A generalized complex structure on (M, H) is a complex structure $\mathbb{J} : \mathbb{T}M \to \mathbb{T}M$ whose $+i$-eigenbundle

$$L = \mathsf{ker}(\mathbb{J} - i1) \subset \mathbb{T}M \otimes \mathbb{C}$$

is a Dirac structure.

Note that the condition that L be maximal isotropic is equivalent to $\mathbb{J}$ being an orthogonal transformation for the canonical metric on $\mathbb{T}M$.

Exercise 3.5. Let $I \in \mathsf{End}(TM)$ such that $I^2 = -1$. Show that

$$\begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}$$

defines a generalized complex structure if and only if I is a usual complex structure and H has type $(2,1) + (1,2)$. □

Exercise 3.6. Let $\omega \in \Omega^2$ be nondegenerate. Show that

$$\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

defines a generalized complex structure if and only if ω is a symplectic form and $H = 0$. □

One of the most intriguing aspects of generalized complex geometry is that it unifies complex and symplectic structures. In general, a generalized complex structure is a hybrid of complex and symplectic structure; to be more precise, if we project L to $TM \otimes \mathbb{C}$, we obtain a complex distribution E. The corresponding real distribution $\Delta = E \cap \overline{E}$ then defines a singular symplectic foliation, deriving from a real Poisson structure Q defined by

$$\{f, g\}_Q = \langle \mathbb{J}(df), dg \rangle.$$

The distribution E then defines an integrable complex structure transverse to the leaves of the above foliation. In this sense, a generalized complex manifold defines a singular symplectic foliation with a transverse holomorphic structure. By now, many interesting generalized complex manifolds are known, especially in dimension 4. See [10, 11] for a construction of generalized complex structures on 4-manifolds which do not admit either complex or symplectic structures. We shall see some nontrivial examples of generalized complex structures in the next section.

3.3 Generalized Kähler geometry

In this section we assume M is a compact manifold equipped with a closed 3-form H, determining an exact Courant algebroid $(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$.

Definition 3.7. A generalized Kähler structure is a pair $(\mathbb{J}_A, \mathbb{J}_B)$ of commuting generalized complex structures, such that the combination

$$-\mathbb{J}_A \mathbb{J}_B = \mathbb{G} \tag{13}$$

defines a generalized Riemannian metric.

The primary example is that of a Kähler structure, in which we take

$$\mathbb{J}_A = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \qquad \mathbb{J}_B = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}.$$

Then these commute because the Kähler condition requires $\omega I + I^* \omega = 0$, and the product

$$-\mathbb{J}_A \mathbb{J}_B = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix},$$

defining a generalized Riemannian metric with $b = 0$.

The fact that $\mathbb{J}_A$ and $\mathbb{J}_B$ commute immediately means that $\mathbb{T}M \otimes \mathbb{C}$ decomposes as a direct sum of the common eigenspaces of this pair of operators. More precisely,

$$\mathbb{T}M \otimes \mathbb{C} = \ell_+ \oplus \ell_- \oplus \overline{\ell_+} \oplus \overline{\ell_-},$$

where the summands are simultaneously eigenbundles for $\mathbb{J}_A$ and $\mathbb{J}_B$, according to the following chart:

$$
\begin{array}{c|ccc}
\mathbb{J}_B & & & \\
+i & \overline{\ell_-} & \ell_+ & \\
-i & \overline{\ell_+} & \ell_- & \\
\hline
 & -i & i & \mathbb{J}_A
\end{array}
$$

That is, if $L_A = \mathsf{ker}(\mathbb{J}_A - i1)$ and $L_B = \mathsf{ker}(\mathbb{J}_B - i1)$, then we have

$$\ell_+ = L_A \cap L_B \qquad \ell_- = L_A \cap \overline{L_B}.$$

Exercise 3.8. Using (13), show that the generalized metric may also be described in terms of these four subbundles:

$$V_+ \otimes \mathbb{C} = \ell_+ \oplus \overline{\ell_+}, \qquad V_- \otimes \mathbb{C} = \ell_- \oplus \overline{\ell_-}. \tag{14}$$

□

Bi-Hermitian structure

By projecting the four subbundles above to the tangent bundle, we show that a generalized Kähler structure may be described equivalently in terms of some recognizable geometric structures on the base manifold.

We already know that V_+ (and hence, V_-) is the graph of $g + b$, where g is a Riemannian metric and $b \in \Omega^2(M, \mathbb{R})$. Since we have the decomposition (14), and since the projection $\pi : V_\pm \to TM$ is an isomorphism, we see that a generalized Kähler structure determines and is determined by a pair of complex structure I_+ (coming from projecting V_+) and I_- (coming from projecting V_-). These are integrable because $\ell_\pm$ are Courant involutive and the projection to TM is bracket-preserving. Since $\ell_\pm$ are isotropic, it follows immediately that $I_\pm$ are compatible with g, defining a pair of Hermitian structures with coinciding metric, or a *bi-Hermitian structure.*

Given the generalized metric $g + b$ and the complex structures $I_\pm$, we have the associated Hermitian 2-forms

$$\omega_\pm = gI_\pm,$$

and we may use these forms to express the subbundles $\ell_\pm$, in the following way.

$$\ell_+ = \{X + (g + b)X \ : \ X \in T^+_{1,0}M\} \tag{15}$$
$$= \{X - i\omega_+(X) + b(X) \ : \ X \in T^+_{1,0}M\} \tag{16}$$

Now that ℓ_+ has been expressed as the partial graph of a 2-form, we may express the involutivity of ℓ_+ in terms of the derivative of this 2-form. Note that ℓ_+ is involutive because it is the intersection $L_A \cap L_B$ of Dirac structures.

In particular, the involutivity condition can be expressed as follows:

$$[e^{b-i\omega_+}X, e^{b-i\omega_+}Y]_H = e^{b-i\omega_+}[X,Y] \qquad \forall X, Y \in C^\infty(T^+_{1,0}M).$$

Now using (4), we see that the above holds if and only if

$$i_X i_Y (H + db - id\omega_+) = 0 \qquad \forall X, Y \in C^\infty(T^+_{1,0}M). \tag{17}$$

Exercise 3.9. Using the above as a model, show that ℓ_- is involutive if and only if

$$i_X i_Y (H + db + id\omega_-) = 0 \qquad \forall X, Y \in C^\infty(T^-_{1,0}M). \tag{18}$$

□

Equation (17) is equivalent to the condition that the $(3,0)$ and $(2,1)$ components of the 3-form $H + db - id\omega_+$ vanish: Letting $H + db = H_0$ for simplicity, this is equivalent to the system:

$$H_0^{3,0} = 0 \tag{19}$$

$$H_0^{2,1} = i\partial\omega_+. \tag{20}$$

Equivalently, H_0 must have type $(2,1) + (1,2)$ with respect to I_+ and ω_+ must satisfy

$$d^c_+\omega_+ = i(\overline{\partial} - \partial)\omega_+ = -H_0.$$

Exercise 3.10. Show that the corresponding conditions for ω_- are that H_0 must be of type $(2,1)+(1,2)$ with respect to I_- and ω_- must satisfy

$$d^c_-\omega_- = H_0.$$

□

The above is the main argument needed for proving the following result.

Theorem 3.11 ([23])**.** *A Generalized Kähler structure $(\mathbb{J}_A, \mathbb{J}_B)$ on the Courant algebroid defined by (M, H) is equivalent to a generalized Riemannian metric $g + b$ and two complex structures $I_\pm$, each compatible with g, such that the corresponding Hermitian forms $\omega_\pm = gI_\pm$ satisfy*

$$-d^c_+\omega_+ = d^c_-\omega_- = H + db. \tag{21}$$

If we mod out by B-field gauge equivalence, then we may eliminate b in the above data.

It is in the bi-Hermitian form above that Gates, Hull, and Roček originally discovered generalized Kähler geometry. Note that Equation 21 implies that $\omega_\pm$ are not closed but rather satisfy the *pluriclosed* condition

$$dd^c_\pm \omega_\pm = 0.$$

This is particularly interesting in the case that the cohomology class $[H]$ is nonzero: then the pluriclosed condition contradicts the ∂-$\overline{\partial}$ Lemma of Kähler geometry, implying that a generalized Kähler manifold with $[H] \neq 0$ cannot admit a Kähler metric. In particular it cannot be a projective algebraic variety.

Generalized Hodge decomposition

Since we have a pair $(\mathbb{J}_A, \mathbb{J}_B)$ of elements of $\mathfrak{so}(\mathbb{T}M)$, they act on differential forms via the spin representation, inducing a $\mathbb{Z} \times \mathbb{Z}$ grading via the common eigenvalues of $\mathbb{J}_A$ amd $\mathbb{J}_B$.

Let $U_{p,q} \subset \Omega^*(M, \mathbb{C})$ consist of the differential forms which are eigenvectors of both $\mathbb{J}_A$ and $\mathbb{J}_B$ with eigenvalues (ip, iq) respectively. In this way we obtain a (p, q) decomposition of the differential forms into the following diamond:

$$\begin{array}{ccccccccc}
 & & & & U_{0,n} & & & & \\
 & & & \cdots & & \cdots & & & \\
 & U_{-n+1,1} & & & & & & U_{n-1,1} & \\
U_{-n,0} & & & & \cdots & & & & U_{n,0} \\
 & U_{-n+1,-1} & & & & & & U_{n-1,-1} & \\
 & & & \cdots & & \cdots & & & \\
 & & & & U_{0,-n} & & & &
\end{array}$$

This decomposition is orthogonal with respect to the Born-Infeld metric, and gives rise to the following decomposition of the exterior derivative:

$$d_H = \delta_+ + \delta_- + \overline{\delta}_+ + \overline{\delta}_-,$$

where the differential operators act as follows:

$$\begin{tikzcd}
U_{p-1,q+1} & & U_{p+1,q+1} \\
& U_{p,q} \arrow[ul, "\delta_-"'] \arrow[ur, "\overline{\delta}_+"] \arrow[dl, "\delta_+"] \arrow[dr, "\overline{\delta}_-"'] & \\
U_{p-1,q-1} & & U_{p+1,q-1}
\end{tikzcd}$$

Using a similar argument as in (12), we easily obtain the relation of adjointness between these operators, which is a generalization of the Kähler identities:

Exercise 3.12 (hi)**.** Prove the generalized Kähler identities

$$\overline{\delta}_+^* = -\delta_+ \quad \text{and} \quad \overline{\delta}_-^* = \delta_- .$$

□

These simple identities imply the equality of all available Laplacians:

$$\Delta_{d_H} = 4\Delta_{\overline{\delta}_\pm} = 4\Delta_{\delta_\pm},$$

and so, finally, we obtain a (p,q) decomposition for the twisted cohomology of any compact generalized Kähler manifold.

Theorem 3.13 (Hodge decomposition)**.** *The twisted cohomology of a compact $2n$-dimensional generalized Kähler manifold carries a Hodge decomposition:*

$$H^\bullet_{d_H}(M,\mathbb{C}) = \bigoplus_{\substack{|p+q|\leq n \\ p+q\equiv n (mod\ 2)}} \mathcal{H}^{p,q},$$

where $\mathcal{H}^{p,q}$ are Δ_{d_H}-harmonic forms in $U_{p,q}$.

Note that in the usual Kähler case, this (p,q) decomposition is *not* the Dolbeault decomposition: it was called the Clifford decomposition by Michelsohn [31], and there is an orthogonal transformation, called the Hodge automorphism, taking it to the usual Dolbeault decomposition. A striking feature of the Clifford decomposition is that a form of type (p,q) is closed if and only if it is co-closed and hence harmonic.

3.4 Examples

There are two main sources of examples of generalized Kähler metrics:

Example 3.14 (The even-dimensional compact semi-simple Lie groups)**.** It has been known since the work of Samelson [34] and Wang [37] that any compact even-dimensional Lie group admits left- and right-invariant complex structures I_+, I_- respectively, and that if the group is semi-simple, these can be chosen to be Hermitian with respect to the bi-invariant metric induced from the Killing form $g = \langle , \rangle$. The idea, then, is to use (g, I_+, I_-) as a bi-Hermitian structure with $b = 0$ and to show that it is integrable with respect to $H(X,Y,Z) = \langle [X,Y], Z\rangle$, the bi-invariant Cartan 3-form.

To see that this works, we compute $d^c_+\omega_+$:

$$\begin{aligned} A = d^c_+\omega_+(X,Y,Z) &= d\omega_+(I_+X, I_+Y, I_+Z) \\ &= -\omega_+([I_+X, I_+Y], I_+Z) + c.p. \\ &= -\langle [I_+X, I_+Y], Z\rangle + c.p. \\ &= -\langle I_+[I_+X, Y] + I_+[X, I_+Y] + [X,Y], Z\rangle + c.p. \\ &= (2\langle [I_+X, I_+Y], Z\rangle + c.p.) - 3H(X,Y,Z) \\ &= -2A - 3H(X,Y,Z), \end{aligned}$$

Proving that $d^c_+\omega_+ = -H$. Since the right Lie algebra is anti-isomorphic to the left, the same calculation with I_- yields $d^c_-\omega_- = H$, and finally we have

$$-d^c_+\omega_+ = d^c_-\omega_- = H,$$

as required. □

The other large source of generalized Kähler structures comes from compact holomorphic Poisson Kähler manifolds. In [24], we find the following result:

Theorem 3.15. *Let (M, I, σ) be a compact holomorphic Poisson manifold, so that $\sigma \in H^0(\wedge^2 T_{1,0}M)$ satisfies $[\sigma, \sigma] = 0$, and let ω be a Kähler form. Then there is an analytic family of generalized Kähler structures $(\mathbb{J}_A(t), \mathbb{J}_B(t))$, for t in a nonempty neighbourhood of $0 \in \mathbb{R}$, which coincides with the given Kähler structure at $t = 0$ and has the property that its underlying bi-Hermitian structure $(I_+(t), I_-(t))$ is such that $I_+(t) = I$ remains constant while $I_-(t)$ evolves in such a way that*

$$-\tfrac{1}{4}[I_+(t), I_-(t)]g^{-1} = t\mathrm{Re}(\sigma),$$

and furthermore, $I_-(t)$ deforms with Kodaira-Spencer class given by $[\sigma(\omega)] \in H^1(T)$.

Using the above result, we obtain nontrivial generalized Kähler deformations of Kähler manifolds which admit a holomorphic Poisson structure. Examples include the complex projective spaces, which have many interesting holomorphic Poisson structures [12].

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