On Serre's Conjecture II

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Given a field K and an algebraic variety X defined over K, it is a central question in arithmetic geometry to determine whether X has *rational points*. From a number theoretic perspective, this question is often asked for some specific fields, for instance number fields or function fields of curves over finite fields, and one tries to relate the strong arithmetic properties of those fields to the behaviour of rational points on varieties defined over them.

In this course, we will choose a less arithmetic and more algebraic approach: we will aim at relating the *algebraic internal properties* of quite general fields to the behaviour of rational points on varieties defined over them. More precisely, given a large class of algebraically defined fields, we would like to determine if there are large classes of varieties defined over them that automatically have rational points.

For that purpose, the first two steps should consist in:

- understanding what we mean by "algebraic internal properties" of fields;
- understanding which "large classes of varieties" are suitable to be studied in this context.

Remark 0.1. These course notes are intended to be understandable by Master students.

1. Galois cohomology

In order to encode the "internal algebraic properties" of fields, we start by introducing Galois cohomology.

Definition 1.1. *Let K be a field and let $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ be its absolute Galois group. A* Galois module *over K is a (discrete) abelian group M endowed with a continuous action of the profinite group G_K such that, for all $g \in G_K$, the map*

$$\begin{aligned} \varphi_g : M &\to M \\ m &\mapsto g \cdot m \end{aligned}$$

is a group homomorphism.

Example 1.2.

1. Any abelian group with the trivial action of G_K.

2. The abelian groups K^{sep}, $K^{\mathrm{sep}\times}$ and $\mu_n(K) := \{x \in K^{\mathrm{sep}\times} : x^n = 1\}$ for n coprime to the characteristic of K are naturally Galois modules over K.

Now set:

$$K^0(G_K, M) := M,$$
$$K^i(G_K, M) := \{\text{continuous functions } f : G_K^i \to M\},$$

and consider the sequence:

$$K^0(G_K, M) \xrightarrow{d^0} K^1(G_K, M) \xrightarrow{d^1} K^2(G_K, M) \xrightarrow{d^2} ...$$

where the coboundary map d^i is defined as follows:

$$d^i f(g_1, ..., g_{i+1}) = g_1 f(g_2, ..., g_{i+1}) + \sum_{j=1}^{i} (-1)^i f(g_1, ..., g_j g_{j+1}, ..., g_{i+1}) + (-1)^{i+1} f(g_1, ..., g_i).$$

One can easily check that it is a complex: $d^i \circ d^{i-1} = 0$ for all i.

Definition 1.3. *We define the* Galois cohomology groups *of M as:*

$$H^0(K, M) := M^{G_K} = \{m \in M : \forall g \in G_K, g \cdot m = m\},$$
$$H^i(K, M) := \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1}) \quad \textit{for } i > 0.$$

Example 1.4.

$$H^1(K, M) = \frac{\{f : G_K \xrightarrow{\mathrm{cont}} M | \forall s, t \in G_K, f(st) = f(s) + s f(t)\}}{\{f : G_K \to M | \exists m \in M, \forall s \in G_K, f(s) = s \cdot m - m\}}.$$

Therefore, if G_K acts trivially on M:

$$H^1(K, M) = \mathrm{Hom}_{\mathrm{cont}}(G_K, M).$$

For instance, $H^1(K, \mathbb{Z})$ is trivial and $H^1(K, \mathbb{Q}/\mathbb{Z})$ is the character group of G_K. This last group encodes important Galois theoretic properties of K.

Galois cohomology enjoys a certain number of nice properties that allow to carry out computations. For instance:

1. **Long exact sequence.** Any short exact sequence of Galois modules:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

induces a natural long exact sequence:

$$\begin{aligned} 0 &\to H^0(K, A) \xrightarrow{f^*} H^0(K, B) \xrightarrow{g^*} H^0(K, C) \\ &\to H^1(K, A) \xrightarrow{f^*} H^1(K, B) \xrightarrow{g^*} H^1(K, C) \\ &\to H^2(K, A) \xrightarrow{f^*} H^2(K, B) \xrightarrow{g^*} ... \end{aligned}$$

2. **Restriction-corestriction.** If L/K is a finite extension, the Galois group G_L is a subgroup of G_K and hence there is a *restriction map*:

$$\mathrm{Res}_{L/K} : H^i(K, M) \to H^i(L, M)$$

for every Galois module M. When $i = 0$, one can also easily construct a *corestriction morphism* in the other direction:

$$\begin{aligned} \mathrm{Cor}_{L/K} : H^0(L, M) &\to H^0(K, M) \\ m &\mapsto \sum_{\sigma \in G_K/G_L} \sigma m \end{aligned}$$

that satisfies $\mathrm{Cor}_{L/K} \circ \mathrm{Res}_{L/K} = [L : K]$. This construction can in fact be extended to higher-degree cohomology to get for each i a morphism:

$$\mathrm{Cor}_{L/K} : H^i(L, M) \to H^i(K, M)$$

such that $\mathrm{Cor}_{L/K} \circ \mathrm{Res}_{L/K} = [L : K]$.

3. **Cohomology of the additive group.** For every $i \geq 1$, the group $H^i(K, K^{\mathrm{sep}})$ is trivial.

4. **Hilbert's Theorem 90.** The group $H^1(K, K^{\mathrm{sep}\times})$ is trivial.

Let us illustrate cohomology groups computations through two simple but important examples.

Example 1.5.

1. **Kummer theory.** Let n be an integer prime to $\mathrm{char}(K)$. We have an exact sequence of Galois modules:

$$1 \to \mu_n(K^{\mathrm{sep}}) \to K^{\mathrm{sep}\times} \to K^{\mathrm{sep}\times} \to 1,$$

hence an exact sequence of abelian groups:

$$K^\times \to K^\times \to H^1(K, \mu_n(K^{\mathrm{sep}})) \to H^1(K, K^{\mathrm{sep}\times}),$$

in which the first arrow is the n-th power map and the last group is trivial by Hilbert's Theorem 90. We deduce that the group $H^1(K, \mu_n(K^{\mathrm{sep}}))$ is isomorphic to $K^\times/K^{\times n}$ and hence encodes information on the multiplication operation on K.

2. **Artin-Schreier.** Let $p := \mathrm{char}(K) > 0$. We have an exact sequence of Galois modules:

$$0 \to \mathbb{Z}/p\mathbb{Z} \to K^{\mathrm{sep}} \xrightarrow{\phi} K^{\mathrm{sep}} \to 0,$$

where $\phi(x) = x^p - x$. We therefore get a long exact sequence:

$$0 \to \mathbb{Z}/p\mathbb{Z} \to K \xrightarrow{\phi} K \to H^1(K, \mathbb{Z}/p\mathbb{Z}) \to H^1(K, K^{\mathrm{sep}}) \to H^1(K, K^{\mathrm{sep}}) \to H^2(K, \mathbb{Z}/p\mathbb{Z}) \to ...$$

Since $H^i(K, K^{\mathrm{sep}}) = 0$ for all $i > 0$, we deduce that $H^1(K, \mathbb{Z}/p\mathbb{Z}) = K/\phi(K)$ and $H^i(K, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i > 1$.

For our purposes, we will focus on an invariant that measures in some sense the complexity of the Galois cohomology of a field and that is called the *cohomological dimension of the field*.

Definition 1.6. *Let K be a field.*

1. *Let p be a prime number. The* p-cohomological dimension $\mathrm{cd}_p(K)$ *of K is the largest integer n such that, for each $r \geq n+1$, for each p-torsion Galois module M over K, the group $H^r(K,M)$ vanishes.*

2. *The* cohomological dimension $\mathrm{cd}(K)$ *of K is the supremum of the $\mathrm{cd}_p(K)$ when p describes all primes numbers. It is also the largest integer n such that, for each $r \geq n+1$, for each torsion Galois module M over K, the group $H^r(K,M)$ vanishes.*

Note that the cohomological dimension of a field only depends on its absolute Galois group. The next examples illustrate the behaviour of the cohomological dimension.

Example 1.7.

1. A field has cohomological dimension 0 if, and only if, K is separably closed.

2. Finite fields have cohomological dimension 1.

3. p-adic fields have cohomological dimension 2. More generally, if K is a complete discrete valuation field with residue field k and if $p \neq \mathrm{char}(k)$, then we have $\mathrm{cd}_p(K) = \mathrm{cd}_p(k) + 1$.

4. If k is a field and p is a prime number different from $\mathrm{char}(k)$, then we have $\mathrm{cd}_p(k(T_1, \ldots, T_n)) = \mathrm{cd}_p(k) + n$.

5. Take $p = \mathrm{char}(K)$. Recall that, according to example 1.5, the group $H^i(K, \mathbb{Z}/p\mathbb{Z})$ is trivial whenever $i > 1$. In fact, one can even prove that $\mathrm{cd}_p(K)$ is always at most 1. Because of this fact, the previous two examples fail when $p = \mathrm{char}(k)$.

The last three examples suggest that our definition of the p-cohomological dimension is not the good definition when $p = \mathrm{char}(K)$. One way to modify that definition consists in using the fppf cohomology. We will however give here a much more elementary and explicit definition in terms of differentials that can be used for computations.

Definition 1.8 (Kato, [Kat82]). *Assume that K has characteristic $p > 0$. Consider the absolute differential module $\Omega^1_{K/\mathbb{Z}}$, defined as the quotient of $\bigoplus_{x \in K} K \cdot dx$ by the relations:*

$$dr,\ r \in \mathbb{Z}; \qquad d(x+y) - dx - dy,\ x, y \in K; \qquad d(xy) - xdy - ydx,\ x, y \in K.$$

Let Ω^i_K be the i-th exterior product over K of the absolute differential module $\Omega^1_{K/\mathbb{Z}}$ and let $H^{i+1}_p(K)$ be the cokernel of the morphism $\mathfrak{p}^i_K : \Omega^i_K \to \Omega^i_K / d(\Omega^{i-1}_K)$ defined by

$$x \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_i}{y_i} \mapsto (x^p - x) \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_i}{y_i} \mod d(\Omega^{i-1}_K),$$

for $x \in K$ and $y_1, \ldots, y_i \in K^\times$. The p-cohomological dimension $\mathrm{cd}_p(K)$ *of K is the smallest integer δ (or ∞ if such an integer does not exist) such that $[K : K^p] \leq p^\delta$ and $H^{\delta+1}_p(L) = 0$ for all finite extensions L of K.*

This definition is modeled so that the following (quite difficult) results hold:

Theorem 1.9. *Let k be a field of characteristic $p > 0$.*

1. *(Kato, [Kat82]) Let K be a complete discrete valuation field with residue field k. Then $\mathrm{cd}_p(K) = \mathrm{cd}_p(k) + 1$.*
2. *(Kato-Kuzumaki, [KK86]) For $n \geq 0$, we have $\mathrm{cd}_p(k(T_1, \dots, T_n)) = \mathrm{cd}_p(k) + n$.*

2. Principal homogeneneous spaces

We are now going to introduce the varieties we will be interested in during this course.

Definition 2.1. *Let K be a field and let G be an algebraic group over K. A* principal homogeneous space *(or a* torsor*) under G is a variety X endowed with a (right) action $f : X \times_K G \to X$ such that $\pi : X \times_k G \to X \times_k X, (x, g) \mapsto (x, x \cdot g)$ is an isomorphism. When K is perfect, this last condition means that $G(K^{\mathrm{sep}})$ acts simply and transitively on $X(K^{\mathrm{sep}})$.*

Example 2.2.

1. An algebraic group G is naturally a torsor under itself: it is called the trivial torsor.
2. Given a positive integer n prime to $\mathrm{char}(K)$ and an element $a \in K^\times$, the equation $x^n = a$ defines a torsor under μ_n.
3. Given a finite separable extension L/K and an element $a \in K^\times$, the equation $N_{L/K}(\mathbf{x}) = a$ defines a torsor under the torus given by the equation $N_{L/K}(\mathbf{x}) = 1$.
4. A smooth projective genus 1 curve is always a torsor under an elliptic curve, its Jacobian.

To simplify, we assume for the time being that K is a perfect field. Let G be an algebraic group over K, let X be some torsor under G and let X_0 be the trivial torsor. One can easily check that X is trivial, that is isomorphic to X_0, if, and only if, X has a rational point. In particular, $X_{K^{\mathrm{sep}}} \cong X_{0,K^{\mathrm{sep}}}$.

It turns out that in algebra there is a very practical method, called *Galois descent*, to classify algebraic objects that become isomorphic over the separable closure of a field. Indeed, let us take K a perfect field, fix some algebraic object X_0 defined over K (for instance a vector space, a K-algebra, a K-variety) and consider all algebraic objects X defined over K such that $X_{K^{\mathrm{sep}}} \cong X_{0,K^{\mathrm{sep}}}$. Such objects are called *twisted forms of* X_0.

Now set $H := \mathrm{Aut}_{K^{\mathrm{sep}}}(X_{0,K^{\mathrm{sep}}})$. The group G_K acts on $X_{0,K^{\mathrm{sep}}}$, and hence it acts on H by conjugation. Given a twisted form X of X_0 together with an isomorphism:

$$\phi : X_{0,K^{\mathrm{sep}}} \to X_{K^{\mathrm{sep}}},$$

one can then easily check that the map:

$$\begin{aligned} f_X : G_K &\to H \\ \sigma &\mapsto \phi^{-1} \circ \sigma(\phi) \end{aligned}$$

satsifies the functional equation:

$$f_X(\sigma\tau) = f_X(\sigma) \circ \sigma(f_X(\tau)).$$

Moreover, if $\phi' : X_{0,K^{\text{sep}}} \to X_{K^{\text{sep}}}$ is another isomorphism and f'_X stands for the map:

$$\begin{aligned} f'_X : G_K &\to H \\ \sigma &\mapsto \phi'^{-1} \circ \sigma(\phi') \end{aligned}$$

then:

$$\forall \sigma \in G_K, f_X(\sigma) = c^{-1} \circ f'_X(\sigma) \circ \sigma(c)$$

where $c = \phi' \circ \phi \in H$. In other words, we have associated to the twisted form X a class $[X]$ in:

$$H^1(K, H) := \frac{\{f : G_K \to H | \forall \sigma, \tau \in G_K, f(\sigma\tau) = f(\sigma) \circ \sigma(f(\tau))\}}{\sim}$$

where $f \sim f'$ if:

$$\exists c \in H, \forall \sigma \in G_K, f(\sigma) = c^{-1} \circ f'(\sigma) \circ \sigma(c).$$

This yields a map:

$$\theta : \frac{\{\text{twisted forms of } X_0\}}{\cong} \to H^1(K, H)$$

sending the class of X_0 to the constant function equal to 1.

Theorem 2.3 (Galois descent)**.** *Endow the sets $\{$twisted forms of $X_0\}/\cong$ and $H^1(K, H)$ with pointed set structures by deciding that the distinguished elements are X_0 and the constant function equal to 1 respectively. The map θ is then a bijection of pointed sets.*

This principle can be applied in many situations. Here are some examples:

1. *Vector spaces.* The automorphism group of the vector space $(K^{\text{sep}})^n$ is $GL_n(K^{\text{sep}})$, hence n-dimensional vector spaces over K are classified up to isomorphism by $H^1(K, GL_n)$. Since there is only one isomorphism class of n-dimensional vector spaces over K, we get $H^1(K, GL_n) = \{1\}$. When $n = 1$, we recover Hilbert's Theorem 90.

2. *Quadratic forms.* The set $H^1(K, O_n)$ classifies non-degenerate quadratic forms over K up to isometry.

3. *Central simple algebras.* A finite-dimensional algebra A over a field K is said to be central simple if its only two-sided ideals are 0 and A itself and its center is reduced to K. According to a Theorem of Wedderburn, this is equivalent to the fact that $A \otimes_K K^{\text{sep}}$ is a matrix algebra $\mathcal{M}_n(K^{\text{sep}})$ for some n. Moreover, the automorphism group of the algebra $\mathcal{M}_n(K^{\text{sep}})$ is $\text{PGL}_n(K^{\text{sep}})$. Hence central simple algebras of dimension n^2 are classified by $H^1(K, \text{PGL}_n)$. Moreover, for each n, we have an exact sequence:

   $$1 \to \mathbb{G}_{\text{m}} \to \text{GL}_n \to \text{PGL}_n \to 1,$$

   hence an exact sequence of pointed sets:

   $$\{1\} = H^1(K, \text{GL}_n) \to H^1(K, \text{PGL}_n) \xrightarrow{\psi_n} H^2(K, K^{\text{sep}\times}).$$

Thanks to the maps $(\psi_n)_{n\geq 1}$, we can associate to each central simple algebra A of any dimension a class $[A]$ in the so-called *Brauer group* $\mathrm{Br}(K) := H^2(K, K^{\mathrm{sep}\times})$. In fact, one can prove that the Brauer group classifies central simple algebras over K up to Brauer equivalence:

$$A \sim B \Leftrightarrow \exists m, n, A \otimes \mathcal{M}_m(K) \cong B \otimes \mathcal{M}_n(K).$$

For more information on central simple algebras, the reader may refer to [GS17].

Coming back to the case of G-torsors for G some algebraic group over K, the automorphism group of the trivial G-torsor over K^{sep} is $G(K^{\mathrm{sep}})$. Hence, Galois descent tells us that torsors under G are classified by the cohomology pointed set $H^1(K, G)$. Of course, the distinguished point in this pointed set corresponds to the trivial torsor.

In the sequel of this course, we will focus on the following more precise version of the questions raised in the introduction of this text: how do rational points on principal homogeneous spaces behave when one works over fields with low cohomological dimension? We start by discussing the case of one-dimensional fields.

3. Serre's Conjecture I

Let us start by an easy observation. Take K a field with $\mathrm{cd}(K) \leq 1$. Let T be a K-torus and let L be a finite separable extension of K that splits T. By Hilbert's Theorem 90, we have $H^1(L, T) = 0$. Hence, by a restriction-corestriction argument, the abelian group $H^1(K, T)$ is killed by $[L : K]$. But, for each integer $n > 0$, we have an exact sequence of Galois modules:

$$1 \to T[n] \to T \xrightarrow{\cdot n} T \to 1$$

and hence a long exact sequence:

$$H^1(K, T[n]) \to H^1(K, T) \xrightarrow{\cdot n} H^1(K, T) \to H^2(K, T[n]) = 0$$

in which $H^2(K, T[n])$ is trivial because K has cohomological dimension ≤ 1. We deduce that the group $H^1(K, T)$ is divisible. Since it has finite exponent, it is in fact trivial.

In other words, every principal homogeneous space under a torus over a field with cohomological dimension ≤ 1 has rational points. It is therefore natural to ask what happens when one replaces the torus T by more general linear groups:

Conjecture 3.1 (Serre's Conjecture I)**.** *Let K be a field with* $\mathrm{cd}(K) \leq 1$*. Then every principal homogeneous space under a reductive group*[1] *has a rational point.*

This conjecture is nowadays a theorem due to Steinberg in 1965 for perfect fields ([Ste65]) and to Borel-Springer in 1968 for general fields ([BS68]).

Sketch of proof. Let G be a reductive group. We want to prove that $H^1(K, G)$ is trivial.

- *Step 1: Case when G is a torus.* This is the particular case we already proved at the beginning of the section!

[1] In these notes, reductive groups are always assumed to be connected.

- *Step 2: Case when G is quasi-split.* According to a Theorem of Springer:

$$H^1(K,G) = \bigcup_{T \subset G} \mathrm{Im}\left(H^1(K,T) \to H^1(K,G)\right),$$

  where T runs over the maximal K-tori of G. Hence $H^1(K,G)$ is trivial.

- *Step 3: General case.* By a twisting argument, one can prove that there exists a quasi-split reductive group H and a bijection between $H^1(K,G)$ and $H^1(K,H)$. Hence $H^1(K,G)$ is trivial.

□

Remark 3.2. In Conjecture 3.1, if one assumes that K is perfect, one can prove that principal homogeneous spaces under an arbitrary connected linear group are trivial (this follows from the triviality of $H^1(K,K^{\mathrm{sep}})$ and from the fact that every unipotent group has a dévissage by additive groups). However, there exist imperfect fields K and connected unipotent groups U over K such that $H^1(K,U)$ is not trivial. For instance, as an exercise, the reader might want to prove that, if $K = \mathbb{F}_p^{\mathrm{sep}}((t))$ for some odd prime p and U is the (connected unipotent) subgroup of $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{a}}$ given by the equation $y^p - y = tz^p$, then $H^1(K,U)$ is not trivial.

We will now move on to the case of two-dimensional fields, which is the core of this mini-course.

4. Quadratic forms and Milnor's Conjecture

Let K be a field with characteristic other than 2. According to a Theorem of Witt, any quadratic form q over K can be written as an orthogonal sum $q = q_{\mathrm{ti}} \perp q_{\mathrm{h}} \perp q_{\mathrm{an}}$ where:

- q_{ti} is *totally isotropic*, i.e. $q_{\mathrm{ti}} = 0$.
- q_{h} is *hyperbolic*, i.e. $q_{\mathrm{h}} \cong \mathbb{H}^m$ for some $m \geq 0$, where $\mathbb{H} = \langle -1, 1\rangle$ is the hyperbolic plane.
- q_{an} is *anisotropic*, i.e. $q_{\mathrm{an}}(v) = 0$ implies $v = 0$.

In order to classify anisotropic quadratic forms over K, one may introduce the Witt ring $W(K)$ of K. As a set, $W(K)$ consists of all anisotropic quadratic forms over K up to isometry. Its addition and its multiplication are defined thanks to the orthogonal sum and the Kronecker's tensor product of quadratic forms:

$$\begin{aligned} q + q' &= (q \perp q')_{\mathrm{an}}, \\ q \cdot q' &= (q \otimes q')_{\mathrm{an}}. \end{aligned}$$

Here, if $q = \langle d_1, \ldots, d_r\rangle$ and $q' = \langle d'_1, \ldots, d'_s\rangle$, then $q \otimes q'$ is the diagonal quadratic form $\langle d_i d'_j : i, j\rangle$.

The set $I(K)$ of even-dimensional anisotropic quadratic forms over K forms an ideal $I(K)$ in $W(K)$ called the *fundamental ideal.* It is generated by the so-called 1-fold Pfister forms:

$$\langle\langle a\rangle\rangle := \langle 1, -a\rangle, \quad a \in K^\times.$$

There is of course an isomorphism given by dimension modulo 2:

$$\begin{aligned} e_0 : W(K)/I(K) &\to \mathbb{Z}/2\mathbb{Z} = H^0(K, \mathbb{Z}/2\mathbb{Z}) \\ q &\mapsto \dim(q) \mod 2. \end{aligned}$$

This is the first invariant one might consider to classify anisotropic quadratic forms.

A second invariant is given by the discriminant $\mathrm{disc}(q)$, that is the determinant in $K^\times/K^{\times 2}$ of the matrix in some basis of q. It induces a group morphism:

$$\begin{aligned} d : I(K) &\to K^\times/K^{\times 2} \\ q &\mapsto (-1)^{\dim q/2}\mathrm{disc}(q). \end{aligned}$$

Now observe that the ideal $I(K)^2$ is spanned 2-fold Pfister forms:

$$\langle\langle a, b\rangle\rangle := \langle\langle a\rangle\rangle \otimes \langle\langle b\rangle\rangle = \langle 1, -a, -b, ab\rangle, \quad a, b \in K^\times,$$

and all these quadratic forms belong to $\mathrm{Ker}(d)$. Hence d induces a morphism:

$$e_1 : I(K)/I(K)^2 \to K^\times/K^{\times 2} = H^1(K, \mathbb{Z}/2\mathbb{Z}),$$

and one can check as an exercise that this is an isomorphism.

These arguments can be continued to construct higher cohomological invariants of quadratic forms. For each $n \geq 0$, one can define a group homomorphism:

$$\begin{aligned} e_n : I(K)^n/I(K)^{n+1} &\to H^n(K, \mathbb{Z}/2\mathbb{Z}) \\ \langle\langle a_1, \dots, a_n\rangle\rangle := \langle\langle a_1\rangle\rangle \otimes \dots \otimes \langle\langle a_n\rangle\rangle &\mapsto f_{a_1,\dots,a_n}, \end{aligned}$$

where:

$$\begin{aligned} f_{a_1,\dots,a_n} : G_K^n &\to \mathbb{Z}/2\mathbb{Z} \\ (g_1, \dots, g_n) &\mapsto \frac{\phi(g_1(\sqrt{a_1}))}{\sqrt{a_1}} \cdots \frac{\phi(g_n(\sqrt{a_n}))}{\sqrt{a_n}} \end{aligned}$$

and ϕ is the isomorphism $\{-1, 1\} \to \mathbb{Z}/2\mathbb{Z}$.

Theorem 4.1 (Milnor's Conjecture [Mil70], proved in [Voe03], [OVV07], [Mor99], [Mor06], [KS00]). *For all $n \geq 0$, the invariant e_n is an isomorphism.*

According to a Theorem of Arason-Pfister (Hauptsatz X.5.1 in [Lam05]), anisotropic quadratic forms in $I(K)^n$ have dimension $\geq 2^n$, and hence the intersection $\bigcap_{n\geq 0} I^n(K)$ is always trivial. In other words, the invariants e_n for $n \geq 0$ form a complete system of invariants that does allow to distinguish non-isometric quadratic forms.

Now let us consider the case when K is a field with cohomological dimension ≤ 2. One then has:

$$I(K)^3 = I(K)^4 = \dots = I(K)^n = \dots = \bigcap_{n\geq 0} I^n(K) = 0.$$

In particular, for any $a, b, c \in K^\times$, the quadratic form $\langle\langle a, b, c\rangle\rangle = \langle 1, -a, -b, ab\rangle \perp c\langle 1, -a, -b, ab\rangle$ is isotropic. But the non-zero elements in the image of $\langle 1, -a, -b, ab\rangle$ form a subgroup of $K^\times$, and hence the equation:

$$x^2 - ay^2 - bz^2 + abt^2 = c$$

automatically has solutions in K. This result turns out to be a very particular case of Serre's Conjecture II.

5. Serre's Conjecture II

Conjecture 5.1 (Serre's Conjecture II [Ser62]). *Let K be a field with $\mathrm{cd}(K) \leq 2$. Then every principal homogeneous space under a semisimple simply connected group has a rational point.*

Recall that any semisimple simply connected group G can be written as a product of Weil restrictions:

$$G = \prod_{i=1}^{n} R_{L_i/K}(H_i)$$

where each H_i is a semisimple simply connected absolutely almost simple group. We then get:

$$H^1(K, G) = \prod_i H^1(L_i, H_i),$$

where each L_i has cohomological dimension ≤ 2. Hence it suffices to prove Serre's Conjecture II for semisimple simply connected absolutely almost simple groups. Such groups are classified by types corresponding to Dynkyn diagrams. There are four families of classical types, that can be described as follows when the field K is perfect:

- A_n: this family contains two subfamilies:
  - ${}^1A_n$: these are groups of the form $\mathrm{SL}(A)$ for A some central simple algebra.
  - ${}^2A_n$: these are groups of the form $\mathrm{SU}(A, \sigma)$ where A is a central simple algebra defined over a quadratic extension of K and endowed with a hermitian involution σ.
- B_n: these are groups of the form $\mathrm{Spin}(V, q)$ where (V, q) is an odd-dimensional non-degenerate quadratic space over K.
- C_n: these are groups of the form $\mathrm{Sp}(A, \sigma)$ where A is an even-dimensional central simple algebra endowed with a symplectic involution σ.
- D_n (non-trialitarian for $n = 4$): these are groups of the form $\mathrm{Spin}(A, \sigma)$ where A is an even-dimensional central simple algebra endowed with an orthogonal involution σ.

For the reader's convenience, we briefly recall some of the objects appearing in this classification. According to a Theorem of Wedderburn, if A stands for a central simple algebra over K, the K^{sep}-algebra $A \otimes_K K^{\mathrm{sep}}$ is a matrix algebra $\mathcal{M}_n(K^{\mathrm{sep}})$. One can then define the reduced norm of A as the composite group homomorphism:

$$\mathrm{Nrd}_A : A^\times \to (A \otimes_K K^{\mathrm{sep}})^\times \cong \mathcal{M}_n(K^{\mathrm{sep}})^\times \to (K^{\mathrm{sep}})^\times,$$

and the group $\mathrm{SL}(A)$ stands for the kernel of the reduced norm.

An *involution* on the central simple algebra A is an additive and anti-multiplicative map $\sigma : A \to A$ such that $\sigma^2 = \mathrm{id}$. It is said to be *of the first kind* if it fixes the center K of A. In that case, $\sigma_{K^{\mathrm{sep}}} = \sigma \otimes \mathrm{id}_{K^{\mathrm{sep}}} : \mathcal{M}_n(K^{\mathrm{sep}}) \cong A \otimes_K K^{\mathrm{sep}} \to A \otimes_K K^{\mathrm{sep}} \cong \mathcal{M}_n(K^{\mathrm{sep}})$ is of the form:

$$m \mapsto g^{-1} \cdot m^t \cdot g$$

for some matrix $g \in \mathrm{GL}_n(K^{\mathrm{sep}})$. The involution σ is said to be orthogonal (resp. symplectic) if g is symmetric (resp. skew-symmetric). When σ is symplectic, the symplectic group $\mathrm{Sp}(A, \sigma)$ is defined as:

$$\mathrm{Sp}(A, \sigma) = \{a \in A : a\sigma(a) = 1\}.$$

Similarly, when σ is orthogonal, the special orthogonal group $\mathrm{SO}(A, \sigma)$ is defined as:

$$\mathrm{SO}(A, \sigma) = \{a \in A : \mathrm{Nrd}_A(a) = 1, a\sigma(a) = 1\}.$$

The spin group $\mathrm{Spin}(A, \sigma)$ is then a μ_2-covering of $\mathrm{SO}(A, \sigma)$:

$$1 \to \mu_2 \to \mathrm{Spin}(A, \sigma) \to \mathrm{SO}(A, \sigma) \to 1.$$

An involution that is not of the first kind is said to be *of the second kind*, *hermitian* or *unitary*. In that case, the elements of A fixed by σ form a subfield of K of degree 2. The special unitary group $\mathrm{SU}(A, \sigma)$ is defined as:

$$\mathrm{SU}(A, \sigma) = \{a \in A : \mathrm{Nrd}_A(a) = 1, a\sigma(a) = 1\}.$$

For more information on involutions on central simple algebras and the properties of the associated groups, the interested reader may refer to [KMRT98].

Of course, there are semisimple simply connected groups that do not lie in the previously mentioned four families of classical types. There are in fact six exceptional types: G_2, F_4, E_6, E_7, E_8 and trialitarian D_4. Almost all articles dealing with Serre's Conjecture II treat different types of groups separately. Some of them aim at proving the conjecture for a given type, while others aim at proving the conjecture for all types over a particular kind of field. In this course, we will mainly focus on the first point of view. However, for the sake of completeness, we first very briefly mention the main known results over specific fields.

5.1 Results in terms of fields

The first main result concerning specific fields deals with the case of complete discrete valuation fields with perfect residue field. This case has been fully solved thanks to the Bruhat-Tits theory:

Theorem 5.2 (Section 4.7 of [BT87])**.** *Let K be a complete discrete valuation field with perfect residue field k. Assume that* $\mathrm{cd}(K) \leq 2$. *Then Serre's Conjecture II holds over K.*

Preuve. See Yisheng Tian's course! □

Of course, fields with more global behaviour have also been studied. In particular, we have the following results:

Theorem 5.3.

(i) ([Kne65a, Kne65b], [Har65, Har66], [Che89]) Totally imaginary number fields satisfy Serre's Conjecture II.

(ii) ([CTGP04], [dJHS11]) Finite extensions of $\mathbb{C}(x, y)$ satisfy Serre's Conjecture II.

(iii) ([CTGP04]) Finite extensions of $\mathbb{C}((x, y))$ satisfy Serre's Conjecture II.

5.2 Known results in terms of groups

From now on, we focus on the study of groups of a given type over arbitrary fields of cohomoplogical dimension ≤ 2. To simplify, unless otherwise stated, all fields in this section will be assumed to be perfect.

5.2.1 Type ${}^1A_n$

Let us start with groups of type ${}^1A_n$. As we have seen, these are groups of the form $\mathrm{SL}(A)$ for A some central simple algebra. In other words, they fit in an exact sequence,

$$1 \to \mathrm{SL}(A) \to \mathrm{GL}(A) \xrightarrow{\mathrm{Nrd}_A} \mathbb{G}_\mathrm{m} \to 1.$$

hence an exact sequence:

$$A^\times \xrightarrow{\mathrm{Nrd}_A} K^\times \to H^1(K, \mathrm{SL}(A)) \to H^1(K, \mathrm{GL}(A)),$$

in which the last term is trivial by Hilbert's Theorem 90. In particular:

$$H^1(K, \mathrm{SL}(A)) \cong K^\times / \mathrm{Nrd}_A(A^\times).$$

For instance, when A is the quaternion algebra:

$$(a, b) = K \oplus Ki \oplus Kj \oplus Kk$$

with relations $i^2 = a$, $j^2 = b$ and $k = ij = -ji$, we have:

$$\mathrm{Nrd}_A(x + yi + zj + tk) = x^2 - ay^2 - bz^2 + abt^2.$$

Principal homogeneous spaces under $\mathrm{SL}(A)$ are then given by:

$$x^2 - ay^2 - bz^2 + abt^2 = c$$

for $c \in K^\times$, and we know by the results of section 4 that they have rational points.

The proof of that result crucially relied on the invariant e_3 of quadratic forms:

$$\begin{aligned} e_3 : I(K)^3/I(K)^4 &\to H^3(K, \mathbb{Z}/2\mathbb{Z}) \\ \langle\langle a, b, c\rangle\rangle &\mapsto f_{a,b,c}, \end{aligned}$$

where $f_{a,b,c}$ is the cocycle:

$$\begin{aligned} f_{a,b,c} &: G_K^3 \to \mathbb{Z}/2\mathbb{Z} \\ (g_1, g_2, g_3) &\mapsto \frac{\phi(g_1(\sqrt{a}))}{\sqrt{a}} \frac{\phi(g_1(\sqrt{b}))}{\sqrt{b}} \frac{\phi(g_n(\sqrt{c}))}{\sqrt{c}} \end{aligned}$$

and ϕ is the isomorphism $\{-1, 1\} \to \mathbb{Z}/2\mathbb{Z}$. The construction of the cocycle $f_{a,b,c}$ is a very particular case of the so-called cup-product in Galois cohomology. Indeed, whenever P and Q are two Galois modules, we have a bilinear cup-product map:

$$\begin{aligned} \cup : H^i(K, P) \times H^j(K, Q) &\to H^{i+j}(K, P \otimes Q) \\ (p, q) &\mapsto p \cup q \end{aligned}$$

where $p \cup q$ is given as a cocycle by:

$$(p \cup q)(g_1, \ldots, g_{i+j}) \mapsto p(g_1, \ldots, g_i) \otimes (g_1 \ldots g_i) \cdot q(g_{i+1}, \ldots, g_{i+j}).$$

The invariant e_3 can then described as:

$$e_3(\langle\langle a, b, c\rangle\rangle) = A \cup c,$$

where $A = (a, b) \in \mathrm{Br}(K)[2] = H^2(K, \mathbb{Z}/2\mathbb{Z})$ and $c \in K^\times/{K^\times}^2 = H^1(K, \mathbb{Z}/2\mathbb{Z})$.

Coming back to the general situation where A is any central simple algebra, denote by n the order of $[A]$ in $\mathrm{Br}(K)$ and assume that n is prime to $\mathrm{char}(K)$. We can then see $[A]$ as an element in $H^2(K, \mu_n)$, and we have a group morphism:

$$\begin{aligned} r_A : H^1(K, \mu_n) &\to H^3(K, \mu_n \otimes \mu_n) =: H^3(K, \mathbb{Z}/n\mathbb{Z}(2)) \\ t &\mapsto [A] \cup t. \end{aligned}$$

By identifying $H^1(K, \mu_n)$ to $K^\times/{K^\times}^n$, one can prove that $\mathrm{Nrd}_A(A^\times)$ is contained in the kernel of r_A, and hence r_A induces an invariant, called Suslin's invariant:

$$R_A : H^1(K, \mathrm{SL}(A)) = K^\times/\mathrm{Nrd}_A(A^\times) \to H^3(K, \mathbb{Z}/n\mathbb{Z}(2)).$$

The following result is a difficult and deep result of Merkurjev and Suslin:

Theorem 5.4 (Th. 24.8 of [Sus85]). *With the previous notations, if $\sqrt{\dim A}$ is square-free, the invariant R_A is injective. In particular, if $\mathrm{cd}(K) \leq 2$, then $H^1(K, \mathrm{SL}(A))$ is trivial and the reduced norm $\mathrm{Nrd}_A : A^\times \to K^\times$ is surjective.*

A quite simple dévissage argument allows to extend the second part of the theorem to the case where $\sqrt{\dim A}$ is any positive integer, and hence allows to settle Serre's Conjecture II for groups of type ${}^1A_n$.

Another more subtle dévissage argument allows to generalize the surjectivity of the reduced norm of central simple algebras over cohomological dimension 2 fields to homogeneous varieties, in particular to varieties that parametrize Borel subgroups inside a given semisimple group. This allows to prove the following generalization of Theorem 5.4, due to Gille:

Theorem 5.5 (Th. 6 of [Gil01]). *Let K be a field with $\mathrm{cd}(K) \leq 2$. Let G be a semisimple simply connected absolutely almost simple group over K and let μ be a finite central k-subgroup of multiplicative type. Then the map:*

$$(G/\mu)(K) \to H^1(K, \mu)$$

induced by the exact sequence $1 \to \mu \to G \to G/\mu \to 1$ is surjective.

When $G = \mathrm{SL}(A)$ and μ is the center of G, we recover Theorem 5.4. This generalization will be crucial to settle Serre's Conjecture II for other classical types.

5.2.2 Type B_n

Recall that groups of type B_n are of the form $\mathrm{Spin}(V, q)$ where (V, q) is an odd-dimensional non-degenerate quadratic space over K. The following result is due to Merkurjev:

Theorem 5.6 (Merkurjev, unpublished)**.** *Serre's Conjecture II holds for groups of type B_n over perfect fields.*

Sketch of proof. Let (V, q) be an odd-dimensional non-degenerate quadratic space over a perfect field K with cohomological dimension ≤ 2. The idea consists in exploiting the exact exact sequence:

$$SO(q)(K) \xrightarrow{\delta_0} K^\times/K^{\times 2} \to H^1(K, \mathrm{Spin}(q)) \xrightarrow{\varphi} H^1(K, SO(q)) \xrightarrow{\delta_1} H^2(K, \mu_2)$$

induced by the short exact sequence:

$$1 \to \mu_2 \to \mathrm{Spin}(q) \to SO(q) \to 1$$

Take $z \in H^1(K, \mathrm{Spin}(q))$. By descent theory, its image in $H^1(K, SO(q))$ corresponds to a quadratic form q' that has same dimension and discriminant as q. Moreover, one can compute the map δ_1 and check that $\delta_1(q') = e_2(q') - e_2(q)$, so that $e_2(q') = e_2(q')$. We deduce that $q - q' \in I(K)^3$. Since K has cohomological dimension ≤ 2, the ideal $I(K)^3$ is trivial. Hence $q = q'$ and $z \in \mathrm{Ker}\varphi$. But according to Theorem 5.5 the map δ_0 is surjective and hence z is trivial. □

5.2.3 Other classical types

Let us now move to the other classical types. The main result is due to Bayer and Parimala:

Theorem 5.7 ([BP95])**.** *Serre's Conjecture II holds for groups of types ${}^2A_n$, C_n and D_n (except trialitary D_4) over perfect fields.*

Sketch of proof. For each type one can use a similar method to the one described in Theorem 5.6. Indeed, if G is a semisimple simply connected absolutely almost simple group of type ${}^2A_n$, C_n or D_n (other than trialitary D_4), it is of the form $\mathrm{SU}(A, \sigma)$, $\mathrm{Sp}(A, \sigma)$ or $\mathrm{Spin}(A, \sigma)$, and the method consists in exploiting the exact sequence:

$$1 \to \mu \to G \to G/\mu \to 1, \tag{1}$$

where μ is the center of G and G/μ is the neutral connected component of the group of automorphisms of A that commute with σ, and in writing the induced cohomological exact sequence:

$$(G/\mu)(K) \xrightarrow{\delta_0} H^1(K, \mu) \to H^1(K, G) \xrightarrow{\varphi} H^1(K, G/\mu) \xrightarrow{\delta_1} H^2(K, \mu). \tag{2}$$

One should then apply descent theory to check that elements in the pointed set $\mathrm{Im}\,(\varphi) = \mathrm{Ker}(\delta_1)$ correspond to certain kinds of unitary involutions on A when G is of type ${}^2A_n$, to conjugacy classes of symplectic involutions on A when G is of type C_n, and to certain kinds of orthogonal involutions on A when G is of type D_n. Finally, in each case, one should prove, by using the assumption that the cohomological dimension of K is at most 2, that such algebraic structures are always trivial. This is the hardest part in the proof, but once such a result is settled, one can deduce that $\mathrm{Im}\,(\varphi) = \mathrm{Ker}(\delta_1)$ is trivial. Moreover, by Theorem 5.5, the map δ_0 is surjective. Hence exact sequence (2) shows that $H^1(K, G)$ is trivial. □

5.2.4 Some exceptional types

Methods similar to the ones we have used to study groups of classical types can in fact also be used to study some exceptional types. Recall that, in order to study Serre's Conjecture II for groups of type ${}^1A_n$, we used the invariant e_3 associated to quadratic forms and Suslin's invariant. These invariants can also be used to prove the following result:

Theorem 5.8 ([BP95])**.** *Serre's Conjecture II holds for groups of types G_2 and F_4 over perfect fields.*

Rough idea of proof. One interprets groups of types G_2 and F_4 as automorphism groups of concrete algebraic structures: the so-called Cayley algebras for type G_2 and the so-called exceptional Jordan algebras for type F_4. By descent theory, this allows to interpret the first cohomology sets of such groups as sets classifying forms of Cayley and Jordan algebras. One then uses quadratic forms invariants and Suslin's invariant to prove that all such forms are trivial when one works on a field with cohomological dimension 2. □

In fact, the e_3 invariant of quadratic forms and Suslin's invariant are particular cases of a much more general construction, called the Rost invariant. More precisely, for each semisimple simply connected group G over a field K, Rost constructed an invariant:

$$R_G : H^1(K, G) \to \varinjlim H^3(K, \mathbb{Z}/n\mathbb{Z}(2)) =: H^3(K, \mathbb{Q}/\mathbb{Z}(2)).$$

The precise construction goes too far for this mini-course since it involves étale cohomology, but the interested reader can refer to [Ser95].

The Rost invariant can for instance be used to study quasi-split groups, that is groups that contain a Borel subgroup defined over K:

Theorem 5.9 ([Gar01], [KMRT98], [Gar10], [Che03], [Gil01])**.** *Let G be a semisimple simply connected quasi-split group over a field K. Assume that G has no E_8 factors. Then Rost's invariant:*

$$H^1(K, G) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2))$$

is injective. In particular, Serre's Conjecture II holds for quasi-split groups of types other than E_8.

The interested reader may want to find some other results about exceptional types in [Gil10] and [Gil19].

5.3 Imperfect fields

We now focus on the case of imperfect fields. Most of the results we have mentioned in the previous sections are known to still hold over imperfect fields:

Theorem 5.10.

(i) ([Gil00]) Serre's Conjecture II holds for groups of type ${}^1A_n$ over imperfect fields.

(ii) ([BFT07]) Serre's Conjecture II holds for groups of types ${}^2A_n$, B_n, C_n and D_n (except trialitary D_4) over imperfect fields.

(iii) ([Gil00]) Serre's Conjecture II holds for quasi-split groups without E_8 factors over imperfect fields.

A natural strategy to try to prove those results could consist in trying to reduce to the characteristic zero case. Indeed, given a torsor Z under a semisimple simply connected group G defined over an imperfect field over K, one could try to find a complete discrete valuation ring B with fraction field $\tilde{K}$ of characteristic zero and residue field K, a semisimple simply connected group scheme $\mathcal{G}$ that lifts G, and a $\mathcal{G}$-torsor $\mathcal{Z}$ that lifts Z. Then one could try to apply Serre's Conjecture II to the generic fiber $\tilde{Z}$ of Z, which is a torsor under a semisimple simply connected group over the characteristic zero field $\tilde{K}$.

However, described in this way, this argument fails. Indeed, the field $\tilde{K}$ has cohomological dimension 3, and hence Serre's Conjecture II cannot be applied. Luckily, for statements (i) and (iii) of the previous theorem, one can modify the argument to make it work. The reason is that the corresponding results over perfect fields (Theorems 5.4 and 5.9) follow from stronger statements that apply to general fields of any cohomological dimension:

- for groups of type ${}^1A_n$, Theorem 5.4 states that Suslin's invariant is always injective for central simple algebras A with $\sqrt{\dim A}$ square-free;
- for quasi-split groups without E_8 factors, Theorem 5.9 states that Rost's invariant is always injective.

One can then use the fact that these statements hold over the cohomological dimension 3 field $\tilde{K}$ to deduce that they also hold over K.

The situation for statement (ii) is very different. Indeed, in this case, over perfect fields, we do not have at our disposal any strengthening of Serre's conjecture II that can be applied to cohomological dimension 3 fields. For that reason, Berhuy, Frings and Tignol's proof cannot rely on a reduction to the characteristic zero case. In fact, it requires to carry out a long, careful and delicate study of involutions of central simple algebras over imperfect fields.

6. Transfer Principles and reduction to characteristic zero fields

Our considerations at the end of the previous section motivate to investigate whether one can prove in some way that Serre's Conjecture II for positive characteristic fields is implied by Serre's Conjecture II for characteristic zero fields. This is the main objective of a recent paper together with Lucchini Arteche ([ILA23]). The main tool to achieve that is provided by what we call *transfer principles* that, given some field with some fixed cohomological dimension, allow to construct other fields with the same or lower cohomological dimension and having some additional properties.

Transfer Principle 1 (From uncountable to countable fields, Prop. 3.1 of [ILA23])**.** *Let K be a field with cohomological dimension δ and let K_0 be a countable subfield. There exists an intermediary field $K_0 \subset K_\infty \subset K$ such that* $\mathrm{cd}(K_\infty) \leq \delta$.

Transfer Principle 2 (From positive to zero characteristic, Th. A of [ILA23])**.** *Let $\tilde{K}$ be a complete discrete valuation field of characteristic* 0 *with countable residue field K*

of cohomological dimension δ. Then there exists a totally ramified extension $\tilde{K}_\dagger/\tilde{K}$ with cohomological dimension δ.

Transfer Principle 3 (From higher to lower cohomological dimension, Th. B of [ILA23])**.** *Let $\delta \geq 1$ be an integer, ℓ a prime number and K a countable field of characteristic 0 and with cohomological dimension δ. Assume that K is ℓ-special, that is every finite extension of K has degree a power of ℓ. For each $x \in K^\times$, there exists an algebraic extension K_x of K that has cohomological dimension $\leq (\delta - 1)$ and such that $x \in N_{L/K}(L^\times)$ for every finite subextension L of K_x/K.*

The proof of Transfer Principle 1 is of a combinatorial nature. The idea is to start with some cohomology class in degree $> \delta$ over K_0 and find a countable extension $K_1 \subset K$ that kills it. Then take a cohomology class of degree $> \delta$ over K_1 and find a countable extension $K_1 \subset K$ that kills it. And so on. One then takes $K_\infty := \bigcup_{i \geq 0} K_i$. For the argument to work, we should kill in this way all cohomology classes of degree $> \delta$ over the K_i's. For that purpose, the difficulty comes from the fact that each time we pass from some K_i to K_{i+1}, we add new cohomology classes that have to be killed. That is why we need, for the argument to work, to carry out a slightly delicate diagonal-like combinatorial argument.

This combinatorial argument is also needed in the more difficult Transfer Principles 2 and 3. For Transfer Principle 2, it needs to be combined with the following key statement:

Proposition 6.1 (Prop. 3.6 of [ILA23])**.** *Let $\tilde{K}$ be a complete discrete valuation field of characteristic 0 with infinite residue field K of characteristic $p > 0$. Let δ be the cohomological dimension of K and let $\tilde{L}/\tilde{K}$ be a finite unramified Galois extension with residue field extension L/K. Assume that $\tilde{K}$ contains a primitive p-th root of unity and that it contains $\sqrt{-1}$ if $p = 2$. Then, for any element $a \in H^{\delta+1}(\tilde{L}, \mathbb{Z}/p\mathbb{Z})$, there exists a finite and totally ramified extension $\tilde{K}'/\tilde{K}$ of p-primary degree such that a is trivial when restricted to $\tilde{K}'\tilde{L}$.*

The proof of this proposition is very long and hard, and requires to use the Bloch-Kato Conjecture as proved by Rost and Voevodsky ([Voe03]) as well as a delicate filtration of the cohomology group $H^{\delta+1}(\tilde{L}, \mathbb{Z}/p\mathbb{Z})$ constructed by Kato ([Kat82]). One big difficulty consists in handling the length of this filtration, which increases when one replaces the field $\tilde{L}$ by the ramified extension $\tilde{K}'\tilde{L}$. Subtle computations in K-theory and with Kähler differentials are involved.

As for Transfer Principle 3, one needs to combine the combinatorial argument of Transfer Principle 1 with the construction of the so-called norm varieties by Suslin and Joukhovitsky in the context of the proof of the Bloch-Kato conjecture ([SJ06]).

Now, in order to reduce Serre's Conjecture II to the case of characteristic zero fields, the main tools are provided by Transfer Principles 1 and 2, as well as a particular case of the Grothendieck-Serre Conjecture that has been proved by Nisnevich (and an extra contribution by Guo) thanks to the Bruhat-Tits theory:

Conjecture 6.2 (Grothendieck-Serre Conjecture, [Ser58], [Gro58], [Gro68a])**.** *Let B be a regular local ring. Let $\mathcal{G}$ be a reductive group scheme over B and let $\mathcal{Z}$ be a $\mathcal{G}$-torsor. If the generic fiber of $\mathcal{Z}$ has a rational point, then $\mathcal{Z}$ has a B-point.*

Theorem 6.3 ([Nis82], [Nis84], [Guo22])**.** *The Grothendieck-Serre Conjecture holds over complete discrete valuation rings.*

We are now ready to state and prove our main application of our Transfer Principles to Serre's Conjecture II:

Corollary 6.4 (Th. E of [ILA23]). *If Serre's conjecture II holds for countable fields of characteristic 0, then it holds for arbitrary fields.*

Remark 6.5. This statement still holds for groups of a given fixed type. In other words, if Λ is some type in the classification of semisimple simply connected groups and if Serre's Conjecture II for groups of type Λ over countable fields of characteristic 0 holds, then it holds for groups of type Λ over arbitrary fields. This provides a new proof of Berhuy, Frings and Tignol's result about groups of classical type over imperfect fields that avoids the delicate study of involutions on central simple algebras over such fields.

Sketch of proof. Take G a semisimple simply connected group over a field K with cohomological dimension ≤ 2 and let Z be a torsor under G. We want to prove that $Z(K) \neq \emptyset$.

- *Step 1.* By Transfer Principle 1, we can find a countable subfield K_∞ of K with cohomological dimension ≤ 2 and such that both G and Z are defined over K_∞. Up to replacing K by K_∞, we may and do assume that K is countable.
- *Step 2.* By general commutative algebra, we can find a complete discrete valuation ring B with fraction field $\tilde{K}$ of characteristic 0 and residue field K. By general results contained in SGA using that G is semisimple simply connected, there exists a semisimple simply connected B-group $\mathcal{G}$ that lifts G and a $\mathcal{G}$-torsor $\mathcal{Z}$ that lifts Z. We denote by $\tilde{G}$ and $\tilde{Z}$ their respective generic fibers.
- *Step 3.* By Transfer Principle 2, we can find a totally ramified extension $\tilde{K}_\dagger$ of $\tilde{K}$ with cohomological dimension ≤ 2. By Transfer Principle 1, we can find a countable subextension $\tilde{K}_{\dagger\infty}$ of $\tilde{K}_\dagger$ over which $\tilde{G}$ and $\tilde{Z}$ are both defined. By assumption $\tilde{Z}(\tilde{K}_{\dagger\infty}) \neq \emptyset$. Hence $\tilde{Z}(\tilde{K}_\dagger) \neq \emptyset$. But by the Grothendieck-Serre Conjecture for complete discrete valuation rings, we deduce that $\mathcal{Z}(\mathcal{O}_{\tilde{K}_\dagger}) \neq \emptyset$. Reducing modulo the maximal ideal of $\mathcal{O}_{\tilde{K}_\dagger}$, we deduce that $Z(K) \neq \emptyset$.

$\square$

The previous proof does not involve Transfer Principle 3. In fact, since Transfer Principle 3 requires to work over ℓ-special fields, it is not well-suited to study rational points but rather 0-cycles of degree 1. In fact, it allows to provide a higher version of Serre's Conjecture II for fields with cohomological dimension ≥ 3:

Theorem 6.6 (Th. D of [ILA23]). *Assume Serre's conjecture II holds for countable fields of characteristic 0. Let K be a field with cohomological dimension $\leq (q+2)$ and let Z be a torsor under a semisimple simply connected group over K. Then:*

$$K_q(K) = \langle N_{L/K}(K_q(L)) : L/K \text{ finite}, Z(L) \neq \emptyset \rangle.$$

Here, $K_q(K)$ stands for the q-th Milnor K-theory group:

$$K_0(K) = \mathbb{Z},$$
$$K_q(K) := \underbrace{K^\times \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} K^\times}_{q \text{ times}} / \langle x_1 \otimes \dots \otimes x_q \mid \exists i, j, i \neq j, x_i + x_j = 1 \rangle,$$

and $N_{L/K} : K_q(L) \to K_q(K)$ is a norm map constructed by Kato (Section 1.7 of [Kat80]).

When $q = 0$, the norm map $N_{L/K} : K_0(L) = \mathbb{Z} \to K_q(K) = \mathbb{Z}$ is multiplication by the degree of the extension L/K, and hence Theorem 6.6 is a weakening of Serre's Conjecture II stating that torsors under semisimple simply connected groups over cohomological dimension 2 fields have points in finite extensions with coprime degree (we say that they have *zero-cycles of degree* 1).

When $q = 1$, the norm map $N_{L/K} : K_1(L) = L^\times \to K_1(K) = K^\times$ is the usual norm in number theory, and hence Theorem 6.6 states that:

$$K^\times = \langle N_{L/K}(L^\times) : L/K \text{ finite}, Z(L) \neq \emptyset \rangle$$

for every torsor Z under a semisimple simply connected group over a cohomological dimension 3 field K.

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