Towards a higher Bruhat-Tits theory

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Let G be a semi-simple Lie group without compact factors, and consider a maximal compact subgroup K in G. For example, we could take $G = \mathrm{SL}_n(\mathbb{R})$ and $K = \mathrm{SO}_n(\mathbb{R})$. The homogeneous space $X := G/K$ is then equipped with a Riemannian metric invariant under G, whose curvature is negative or non-positive. This makes it a Riemannian symmetric space, and studying the action of G on this space often yields deep results about the structure of the group G (e.g., [Mau09], [Par09]).

Let us now move to a non-Archimedean setting. Let $\mathbb{K}$ be a field equipped with a discrete valuation $\omega : \mathbb{K}^\times \to \mathbb{Z}$, and let $\mathbf{G}$ be a connected reductive $\mathbb{K}$-group. Similarly to the case of real Lie groups, it is natural to look for a metric space X with good geometric properties and on which $\mathbf{G}(\mathbb{K})$ acts by isometries in a very transitive manner. Such a space should play a role analogous to symmetric spaces in this non-Archimedian context.

A simple example is given by the case $\mathbf{G} = \mathrm{SL}_2$. More precisely, let $\mathbb{O}$ be the valuation ring in $\mathbb{K}$, and consider the vector space $V = \mathbb{K}^2$. An $\mathbb{O}$-lattice in V is an $\mathbb{O}$-submodule of the form $\mathbb{O}e_1 \oplus \mathbb{O}e_2$ for a certain $\mathbb{K}$-basis (e_1, e_2) of V. Consider the set X_0 of $\mathbb{O}$-lattices in V, up to homothety. The group $\mathbf{G}(\mathbb{K})$ acts naturally on this space. Moreover, X_0 can be equipped with a $\mathbf{G}(\mathbb{K})$-invariant distance d. Indeed, given two elements $x, x' \in X_0$, one can always find two $\mathbb{O}$-lattices L and L' representing them such that $L' \subseteq L$ and $L/L' \cong \mathbb{O}/a\mathbb{O}$ for some $a \in \mathbb{K}^\times$. One then defines $d(x, x') := \omega(a)$.

From a purely combinatorial perspective, X_0 is a good candidate to replace symmetric spaces in this context. However, topologically, it is not interesting as it has the discrete topology. To address this issue, we introduce the non-oriented metric graph X, whose vertex set is X_0 and where two vertices are connected by an edge if they are at distance 1. It can then be shown that X is an infinite tree without leaves. Furthermore, the action of $\mathbf{G}(\mathbb{K})$ on X_0 naturally extends to X, and it is transitive on the set of edges.

To better understand the properties of this action, we now introduce the notion of an apartment. Suppose $\mathbb{K}$ is complete, and define an apartment in X as a subspace that is isometric to $\mathbb{R}$. Apartments then cover the entire space X, and any two points in X belong to a common apartment. Moreover, by identifying each apartment with $\mathbb{R}$, the restrictions of the actions of elements in $\mathbf{G}(\mathbb{K})$ to the apartments are affine transformations, and the group $\mathbf{G}(\mathbb{K})$ acts transitively not only on the set of edges but also on pairs (e, A) where e is an edge and A is an apartment containing e. One says that the action of $\mathbf{G}(\mathbb{K})$ on X is **strongly transitive**.

The space X introduced above is a particular case of what is called a **Euclidean building**. To define this concept, consider a Euclidean space V of dimension d and a finite subgroup W of its isometry group $\mathrm{Isom}(V)$. A hyperplane in V is called a **wall** if it is the set of points fixed by a reflection in W. A connected component of the complement in V of the union of all walls is called a **Weyl (vectorial) chamber**.

Consider now a Euclidean affine space $\mathbb{A}$ with underlying space V. The group $\mathrm{Isom}(\mathbb{A})$ of affine isometries of $\mathbb{A}$ then identifies with the semi-direct product $\mathrm{Isom}(V) \ltimes V$. Let W^{aff} be a subgroup of $\mathrm{Isom}(\mathbb{A})$ whose vector part coincides with W, and such that $W^{\mathrm{aff}} = \mathrm{Stab}_{W^{\mathrm{aff}}}(x) \cdot T$ for some point $x \in \mathbb{A}$ and some subgroup $T \subset V$. A **Weyl (affine) chamber** is any subset of $\mathbb{A}$ of the form $x + C$, where x is a point in $\mathbb{A}$ and C is a Weyl vectorial chamber.

Definition 0.1 (Parreau, [Parr00, Sec. II.1.2])**.** *Let $\mathcal{I}$ be a set and $\mathscr{A} = \{f : \mathbb{A} \to \mathcal{I}\}$ a family of injective applications, whose images are called apartments. We say that $\mathcal{I}$ is a Euclidean building of type $(\mathbb{A}, W^{\mathrm{aff}})$ if the following axioms are true:*

(I1) For all $f \in \mathscr{A}$ and all $w \in W^{\mathrm{aff}}$, $f \circ w \in \mathscr{A}$.

(I2) For all $f, g \in \mathscr{A}$, the set $D := g^{-1}(f(\mathbb{A}))$ is convex in $\mathbb{A}$, and there exists $w \in W^{\mathrm{aff}}$ such that $(g^{-1} \circ f)|_D = w|_D$.

(I3) For any pair of points in $\mathcal{I}$, there exists an apartment containing both.

These axioms allow to define a map $d_{\mathcal{I}} : \mathcal{I} \times \mathcal{I} \to \mathbb{R}_{\geq 0}$ as follows: if x and y are two elements in $\mathcal{I}$, we choose $f \in \mathscr{A}$ such that x and y are in $f(\mathbb{A})$, then $d_{\mathcal{I}}(x, y) := d_{\mathbb{A}}(f^{-1}(x), f^{-1}(y))$ (where $d_{\mathbb{A}}$ denotes the Euclidean distance on $\mathbb{A}$). This construction does not depend on the choices made.

(I4) If C_1 and C_2 are two affine Weyl chambers of $\mathbb{A}$, and f_1 and f_2 are two elements of $\mathscr{A}$, there exist Weyl subchambers $C_1' \subseteq C_1$ and $C_2' \subseteq C_2$ such that $f_1(C_1')$ and $f_2(C_2')$ are contained in the same apartment.

(I5) For every apartment A and every $x \in A$, there exists a map $\rho_{A,x} : \mathcal{I} \to A$ such that $\rho_{A,x}|_A = \mathrm{id}_A$, $\rho_{A,x}^{-1}(\{x\}) = \{x\}$, and $d_{\mathcal{I}}(\rho_{A,x}(y), \rho_{A,x}(z)) \leq d_{\mathcal{I}}(y, z)$ for all $y, z \in \mathcal{I}$.

These axioms automatically impose that $d_{\mathcal{I}}$ is a metric on $\mathcal{I}$.

In two monumental articles of the 1970's and 1980's ([BT72], [BT84]), Bruhat and Tits successfully associated to every reductive group $\mathbf{G}$ defined over a henselian $\mathbb{Z}$-valued field $\mathbb{K}$ a suitable Euclidean building $\mathcal{I}(\mathbf{G})$ endowed with an action of $\mathbf{G}(\mathbb{K})$ by isometries with affine restrictions to apartments and with good transitivity properties. This theory then yielded numerous arithmetic applications. Without aiming to be exhaustive, notable examples include the study of arithmetic subgroups of reductive groups defined over global fields and their properties of cohomological finiteness (e.g., [AB08, Chap. 13]), the study of principal homogeneous spaces under linear groups over local rings (e.g., [Nis84], [Guo20a], [Guo20b]), and the study of representations of p-adic groups via harmonic analysis (e.g., [Sat63], [Mac71], [Bor76], [DeB04], [DeB05], [Sch96]).

The aim of this course is to present the article [HIL20] in collaboration with Hébert and Loisel, in which we seek to generalize the Bruhat-Tits theory to the case of quasi-split reductive groups defined over fields equipped with higher rank valuations. More specifically, given a totally ordered abelian group Λ and a field $\mathbb{K}$ equipped with a Λ-valuation $\omega : \mathbb{K}^\times \to \Lambda$, can we associate to each $\mathbb{K}$-reductive group $\mathbf{G}$ a "higher building" $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ on which $\mathbf{G}(\mathbb{K})$ acts suitably?

This question has been studied in specific instances several times in the literature. For example, in 1984, in the paper [MS84], Morgan and Shalen focused on the case $\mathbf{G} = \mathrm{SL}_2$.

They associated to this group a space that they called a Λ-tree, and they made $\mathrm{SL}_2(\mathbb{K})$ act on it. They then used this construction to study some compactifications of Teichmüller spaces.

In 1994, in the papers [Par94] and [Par00], Parshin studied the case where $\Lambda = \mathbb{Z}^n$ (with the lexicographic order) and $\mathbf{G} = \mathrm{SL}_d$ or PGL_d for arbitrary n and d. He associated to these groups some spaces that are naturally equipped with an action of $\mathbf{G}(\mathbb{K})$ and that share several properties with standard buildings.

Also in 1994, Bennett introduced in his thesis [Ben94] a vast generalization of the concept of Λ-tree: Λ-buildings. In this abstract notion of higher building, apartments are no longer modeled on the affine geometry of the space $\mathbb{R}^k$ for some k but rather on the space Λ^k endowed with the topology induced by the order on Λ. Higher buildings turn out to satisfy a number of axioms similar to axioms (I1)-(I5). As an example, Bennett generalized the constructions of Morgan and Shalen and associated a Λ-building to the group SL_d for any Λ and d.

Since then, several authors have studied and/or used Bennett's Λ-buildings. For instance, without aiming to be exhaustive:

- Schwer and Struyve constructed new families of Λ-buildings using a base change functor ([SS12]),
- Bennett and Schwer provided various different axiomatic frameworks for Λ-buildings ([BS14]),
- Kapranov used Parshin's constructions to study certain Hecke algebras ([Kap01]),
- Kramer and Tent gave a new proof of the Margulis conjecture (which states that any quasi-isometry of a symmetric space of the non-compact type and without rank 1 factors remains at bounded distance from an isometry) by associating Λ-buildings to semi-simple Lie groups ([KT04], [KT09]).

The main Theorem of [HIL20] aims at generalizing the earlier constructions of Λ-buildings associated to SL_d and PGL_d by Morgan and Shalen, Parshin, and Bennett. Indeed, following an approach parallel to that of Bruhat and Tits' theory, we show that to every quasi-split reductive group $\mathbf{G}$ defined over a Henselian valued field $(\mathbb{K}, \omega)$, one can associate a higher building $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ in the sense of Bennett, and that one can endow it with a natural action of the group $\mathbf{G}(\mathbb{K})$. We further prove that $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ is automatically equipped with a fibered structure that helps to understand its geometry.

Let us start by the simplest possible example: the case of the special linear group over a $\mathbb{Z}^2$-valued field.

1. The lattice building of the special linear group

Let $\mathbb{K}$ be a field equipped with a surjective valuation $\omega : \mathbb{K} \to \mathbb{Z}^2 \cup \{+\infty\}$, where $\mathbb{Z}^2$ is endowed with the lexicographic order. Let $\mathbb{O}$ be the ring of integers in $\mathbb{K}$, that is the subring of $\mathbb{K}$ given by elements with non-negative valuation. It is a local ring, whose maximal ideal $\mathbb{M}$ is given by those elements in $\mathbb{O}$ that have positive valuation. The residue field is the quotient $\kappa := \mathbb{O}/\mathbb{M}$.

Example 1.1. Take $\mathbb{K} := k(\!(u)\!)(\!(t)\!)$ for some base field k. One can endow this field with the valuation $\omega : \mathbb{K}^\times \to \mathbb{Z}^2$ defined by

$$\omega\left(\sum_{n\geq N}\sum_{m\geq M_n} a_{m,n}u^m t^n\right) = (n_0, m_0)$$

where:

$$n_0 = \min\{n : \exists m \geq M_n, a_{m,n} \neq 0\},$$
$$m_0 = \min\{m : a_{m,n_0} \neq 0\}.$$

In that way, we have $\omega(t^a u^b) = (a, b)$. The ring of integers $\mathbb{O}$ is then $k[\![u]\!] \oplus tk(\!(u)\!)[\![t]\!]$. Its maximal ideal is:

$$\mathbb{M} = u\mathbb{O} = uk[\![u]\!] \oplus tk(\!(u)\!)[\![t]\!]$$

and the residue field is the field $\kappa = k$.

Given a positive integer ℓ, fix an $(\ell+1)$-dimensional $\mathbb{K}$-vector space V. An **$\mathbb{O}$-lattice** in V is an $\mathbb{O}$-submodule of V of the form $\mathbb{O}b_0 \oplus ... \oplus \mathbb{O}b_\ell$ for some $\mathbb{K}$-basis $(b_0, ..., b_\ell)$ of V. If L_1 and L_2 are two $\mathbb{O}$-lattices in V, we say that they are **homothetic** if there exists $a \in \mathbb{K}^\times$ such that $L_2 = aL_1$. In that case, we denote $L_1 \sim L_2$. Let $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ be the set of $\mathbb{O}$-lattices in V modulo the homothety relation. We say that $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ is the **lattice $\mathbb{Z}^2$-building** of $(\mathrm{SL}(V), \omega)$. The class of an $\mathbb{O}$-lattice V in $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ is denoted $[L]$.

Given two $\mathbb{O}$-lattices L_1 and L_2 such that $L_2 \subset L_1$, we can always find $a_0, \ldots, a_\ell \in \mathbb{O}$ such that $L_1/L_2 \simeq \mathbb{O}/a_0\mathbb{O} \oplus ... \oplus \mathbb{O}/a_\ell\mathbb{O}$. We write $L_2 \leq L_1$ when at least one of the a_i's is a unit in $\mathbb{O}$. In other words, $L_2 \leq L_1$ if, and only if, $L_2 \subset L_1$ and $L_1/L_2 \simeq \mathbb{O}/a_1\mathbb{O} \oplus ... \oplus \mathbb{O}/a_\ell\mathbb{O}$, for some $a_1, ..., a_\ell \in \mathbb{O}$. In that case, the ℓ-tuple $(\omega(a_1), ..., \omega(a_\ell))$ is uniquely determined up to permutation, thus $d(L_1, L_2) := \max\{\omega(a_1), ..., \omega(a_\ell)\} \in \mathbb{Z}^2$ is well-defined. Since $a_1, ..., a_\ell$ are in $\mathbb{O}$, we have $d(L_1, L_2) \geq 0$.

Now, given $x_1, x_2 \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$, one can easily check that there exist $L_1 \in x_1$ and $L_2 \in x_2$ such that $L_2 \leq L_1$. Then $d(x_1, x_2) := d(L_1, L_2) \in \mathbb{Z}^2$ does not depend on the choices of L_1 and L_2, and the map $d : \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega) \to \Lambda$ is a Λ-valued distance on $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ in the following sense:

(D1) *(Positivity)* For all $x, y \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$, we have $d(x, y) \geq 0$;

(D2) *(Separation)* For all $x, y \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$, $d(x, y) = 0$ if and only if $x = y$;

(D3) *(Symmetry)* For all $x, y \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$, we have $d(x, y) = d(y, x)$;

(D4) *(Triangle inequality)* For all $x, y, z \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

1.1 The projection $\pi^{\mathcal{L}}$

Let $\omega_1 : \mathbb{K} \to \mathbb{Z} \cup \{\infty\}$ be the composite of the valuation ω followed by the projection $\mathbb{Z}^2 \to \mathbb{Z}$ on the first coordinate. It is a $\mathbb{Z}$-valuation on $\mathbb{K}$. We may then define:

- the associated ring of integers $\mathcal{O}$, that is those elements in $\mathbb{K}$ whose ω_1-valuation is non-negative;

- the maximal ideal $\mathcal{M}$ of $\mathcal{O}$, which consists of those elements in $\mathcal{O}$ whose ω_1-valuation is positive;
- the residue field $\mathcal{K}_1 := \mathcal{O}/\mathcal{M}$ of $\mathcal{O}$;
- the ring $\mathcal{O}_1 := \mathbb{O}/\mathcal{M}$ and its maximal ideal $\mathcal{M}_1 := \mathbb{M}/\mathcal{M}$.

The ring $\mathcal{O}_1$ is then a valuation ring in $\mathcal{K}_1$ with valuation group:

$$\mathcal{K}_1^\times/\mathcal{O}_1^\times \cong \mathbb{Z}$$

and with residue field:

$$\mathcal{O}_1/\mathcal{M}_1 \cong \mathbb{O}/\mathbb{M} \cong \kappa.$$

Example 1.2. We continue example 1.1. Then ω_1 is the t-adic valuation on $\mathbb{K}$ and we have:

$$\mathcal{O} = k((u))[\![t]\!], \qquad \mathcal{M} = tk((u))[\![t]\!], \qquad \mathcal{K}_1 = k((u)),$$
$$\mathcal{O}_1 = k[\![u]\!], \qquad \mathcal{M}_1 = uk[\![u]\!], \qquad \mathcal{O}_1/\mathcal{M}_1 = k.$$

Now recall that $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ (resp. $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega_1)$) stands for the set of $\mathbb{O}$-lattices of V (resp. the set of $\mathcal{O}$-lattices of V) up to homothety, and consider the map $\pi^{\mathcal{L}} : \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega) \to \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega_1)$ defined by $\pi([L]) = [\mathcal{O}.L]$.

Theorem 1.3.

(i) The map $\pi^{\mathcal{L}}$ is surjective and $\mathrm{SL}(V)$-equivariant.

(ii) For any $\mathcal{O}$-lattice L of V, the stabilizer of the fiber $(\pi^{\mathcal{L}})^{-1}([L])$ is $\mathrm{SL}(L)$. Moreover, the action of $\mathrm{SL}(L)$ on $(\pi^{\mathcal{L}})^{-1}([L])$ factors through $\mathrm{SL}(L/\mathcal{M}L)$.

(iii) Let ω_0 be the valuation of $\mathcal{K}_1$. For any $\mathcal{O}$-lattice L of V, there is an $\mathrm{SL}(L/\mathcal{M}L)$-equivariant bijection Res_L between $(\pi^{\mathcal{L}})^{-1}([L])$ and the lattice $\mathbb{Z}$-building $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(L/\mathcal{M}L), \omega_0)$ of $(\mathrm{SL}(L/\mathcal{M}L), \omega_0)$.

Figures 1 and 2 represent the projection $\pi^{\mathcal{L}}$ for SL_2 and SL_3.

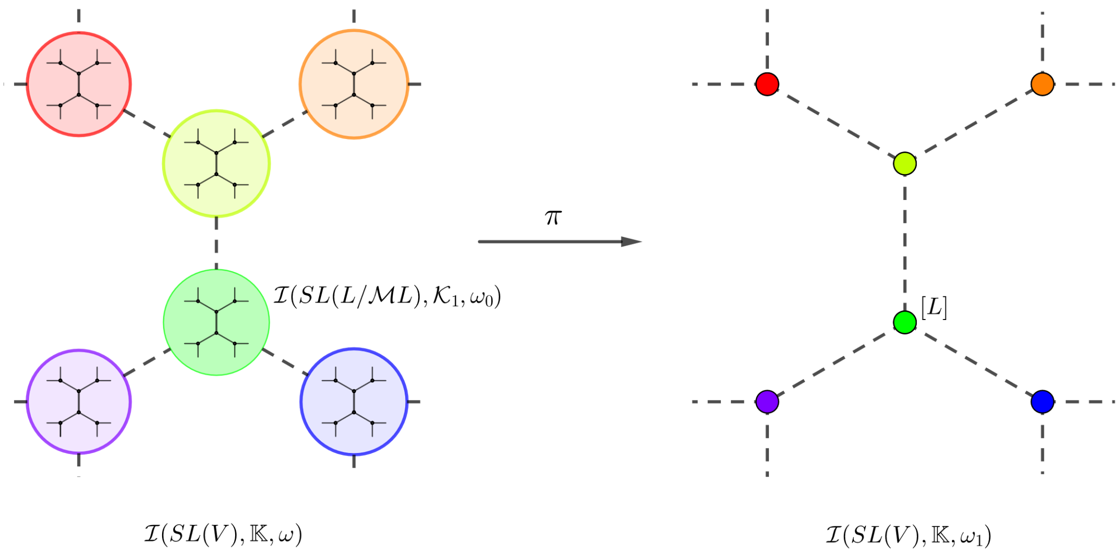

Figure 1: The building of $\mathrm{SL}(V)$ when V is a 2-dimensional vector space over a field $\mathbb{K}$ endowed with a valuation $\omega : \mathbb{K}^\times \to \mathbb{Z}^2$.

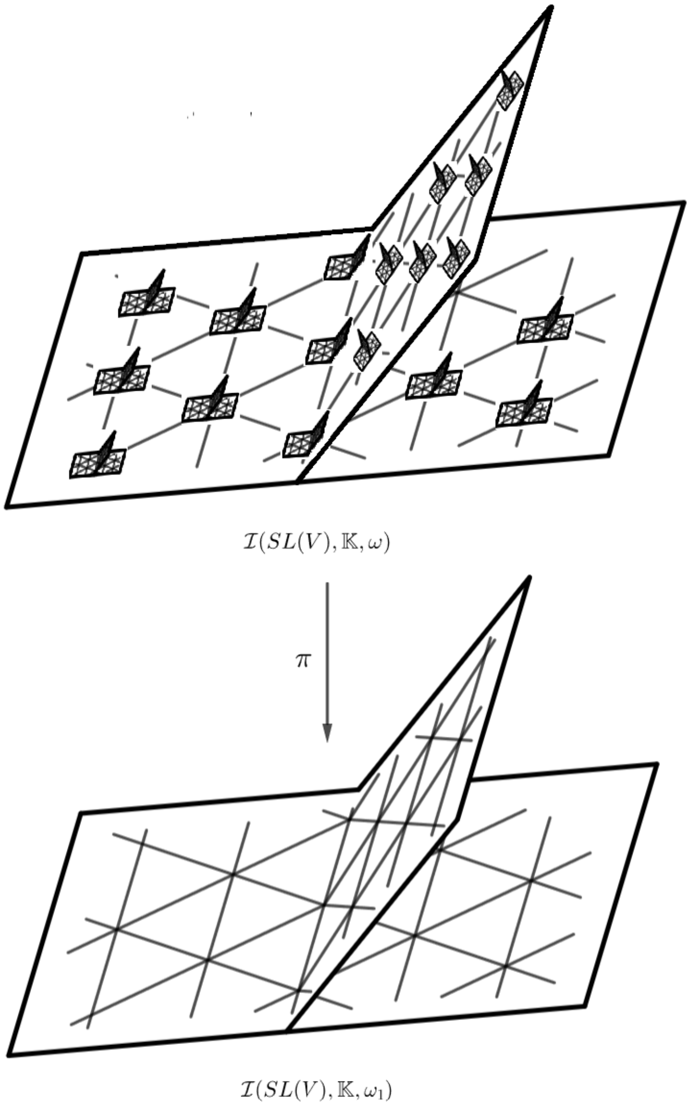

Figure 2: The building of $\mathrm{SL}(V)$ when V is a 3-dimensional vector space over a field $\mathbb{K}$ endowed with a valuation $\omega : \mathbb{K}^\times \to \mathbb{Z}^2$.

As in the classical setting, one can endow the space $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ with an apartment system in the following. We call a **lattice apartment** in $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ a set of the form $\{\mathbb{O}e_0 \oplus \mathbb{O}x_{\lambda_1}e_1 \oplus \cdots \oplus \mathbb{O}x_{\lambda_\ell}e_\ell] \mid \lambda_1, \ldots, \lambda_\ell \in \mathbb{Z}^n\}$ for some basis $(e_0, e_1, \ldots, e_\ell)$ of V and some family $(x_\lambda)_{\lambda \in \mathbb{Z}^n}$ in $\mathbb{K}$ such that $\omega(x_\lambda) = \lambda$ for each λ. Of course, $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ is covered by its apartments and the action of $\mathrm{SL}(V)$ on $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ preserves the apartment system.

In order to get a better understanding of this apartment system, let us carefully study the case of SL_2.

1.2 The apartment system for SL_2

From now on, we assume that $\ell = 1$. We are then interested in the apartment system of the lattice $\mathbb{Z}^2$-tree of $\mathrm{SL}_2 \cong \mathrm{SL}(V)$ over $\mathbb{K}$.

For that purpose, we are going to glue the different fibers of $\pi^{\mathcal{L}}$. For $P \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega_1)$, the fiber $(\pi^{\mathcal{L}})^{-1}(P)$ will be called the **infinitesimal subtree** of $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ above P and will be denoted T_P.

Let $\partial_1 \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ be the set of $\mathbb{O}$-submodules of V of the form $\mathbb{O}b_1 \oplus \mathcal{O}b_2$, where (b_1, b_2) is a $\mathbb{K}$-basis of V, quotiented by the homothety relation.

Definition 1.4. *Fix two elements t and u in $\mathbb{K}$ with respective valuations $(1,0)$ and $(0,1)$.*

*(i) Let $P \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega_1)$. A **ray** of T_P is a sequence of the form $([\mathbb{O}b_1 \oplus \mathbb{O}u^n b_2])_{n \in \mathbb{Z}_{\geq 0}} \in T_P^{\mathbb{Z}_{\geq 0}}$, for some basis (b_1, b_2) of V. We say that two rays $(P_n)_{n \in \mathbb{Z}_{\geq 0}}$ and $(Q_n)_{n \in \mathbb{Z}_{\geq 0}}$ satisfy $(P_n) \sim (Q_n)$ if there exists $k \in \mathbb{Z}$ such that $P_{n+k} = Q_n$ for all $n \gg 0$. A class of rays for this relation is called an **end** of T_P. We denote by $\mathcal{E}(T_P)$ the set of ends of T_P and we set $\mathcal{E}(\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)) = \bigcup_{P \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V),\omega)} \mathcal{E}(T_P)$.*

*(ii) Let $\epsilon \in \{-, +\}$ and set $\eta(-) = 1$ and $\eta(+) = 0$. Let P be an element in $\partial_1 \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ and take $E \in \mathcal{E}(\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega))$. We say that E **converges** to P^ϵ if there exists a $\mathbb{K}$-basis (b_1, b_2) of V such that the ray $([\mathbb{O}b_1 \oplus \mathbb{O}u^{-\epsilon n} b_2])$ represents E and $P = [\mathbb{O}b_1 \oplus \mathcal{O}t^{\eta(\epsilon)} b_2]$. This definition is inspired by the fact that $\bigcap_{n \in \mathbb{Z}_{\geq 0}} u^n \mathbb{O} = t\mathcal{O}$ and $\bigcup_{n \in \mathbb{Z}_{\geq 0}} u^{-n} \mathbb{O} = \mathcal{O}$.*

Remark 1.5. One has:

$$[\mathbb{O}e_1 \oplus \mathbb{O}u^m e_2] \underset{m \to +\infty}{\to} [\mathbb{O}e_1 \oplus \mathcal{O}te_2]^-,$$
$$[\mathbb{O}e_1 \oplus \mathbb{O}u^m e_2] = [\mathbb{O}u^{-m} e_1 \oplus \mathbb{O}e_2] \underset{m \to +\infty}{\to} [\mathcal{O}e_1 \oplus \mathbb{O}e_2]^+.$$

The following proposition shows that limits are uniquely defined:

Proposition 1.6. *Let $\epsilon \in \{+, -\}$ and let E be an element in $\mathcal{E}(\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$. Then there exists a unique element $P \in \partial_1 \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ such that $E \to P^\epsilon$.*

This allows to introduce two maps:

$$\lim{}^+ : \mathcal{E}(\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)) \to \partial_1 \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega),$$
$$\lim{}^- : \mathcal{E}(\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)) \to \partial_1 \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega),$$

that send each $E \in \mathcal{E}(\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ to the unique element $P \in \partial_1 \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)$ such that $E \to P^+$ and $E \to P^-$ respectively. One can then prove the following theorem:

Theorem 1.7. *Let $P \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega_1)$ and set:*

$$\mathcal{S}(P, 1) := \{Q \in \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega_1) |\ d(P, Q) = 1\}.$$

Consider the map:

$$\begin{aligned} \pi_1^{\mathcal{L}} : \partial_1 \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega) &\to \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega_1) \\ [L] &\mapsto [\mathcal{O}L]. \end{aligned}$$

Then:

1. *the map $\lim^+ : \mathcal{E}(T_P) \to (\pi_1^{\mathcal{L}})^{-1}(P)$ is a bijection,*

2. *the map* $\pi^{\mathcal{L}} \circ \lim^- : \mathcal{E}(T_P) \to \mathcal{S}(P,1)$ *is a bijection,*
3. *for all* $E \in \mathcal{E}(T_P)$*, there exists a unique* $\tilde{E} \in \mathcal{E}(\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V),\omega))$ *such that* $\lim^- E = \lim^+ \tilde{E}$*. Moreover,* $\lim^+ E = \lim^- \tilde{E}$ *and* $\tilde{E} \in \mathcal{E}(T_{\pi^{\mathcal{L}}(\lim^- E)})$*.*

One can then decide to glue each end $E \in \mathcal{E}(\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V),\omega))$ with the unique end $\tilde{E} \in \mathcal{E}(\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V),\omega))$ such that $\lim^- E = \lim^+ \tilde{E}$.

Remark 1.8. A good way to understand this glueing involves the notion of border at infinity introduced by Tzu-Jan Li in his lectures. Take P and Q two adjacent vertices in $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V),\omega_1)$. The infinitesimal tress T_P and T_Q are both isomorphic to the lattice building of SL_2 over the field $\mathcal{K}_1$ and hence their borders at infinity $\partial_\infty T_P$ and $\partial_\infty T_Q$ are both isomorphic to $\mathbb{P}^1(\mathcal{K}_1)$. Moreover, the sets of neighbours $\mathcal{S}(P,1)$ and $\mathcal{S}(Q,1)$ are also both isomorphic to $\mathbb{P}^1(\mathcal{K}_1)$. We therefore glue T_P and T_Q by gluing the point in $\partial_\infty T_P$ corresponding to $Q \in \mathcal{S}(P,1)$ with the point in $\partial_\infty T_Q$ corresponding to $P \in \mathcal{S}(Q,1)$.

Once the previous glueing is done, apartments are maximal paths in $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V),\omega)$, as illustrated in figure 3.

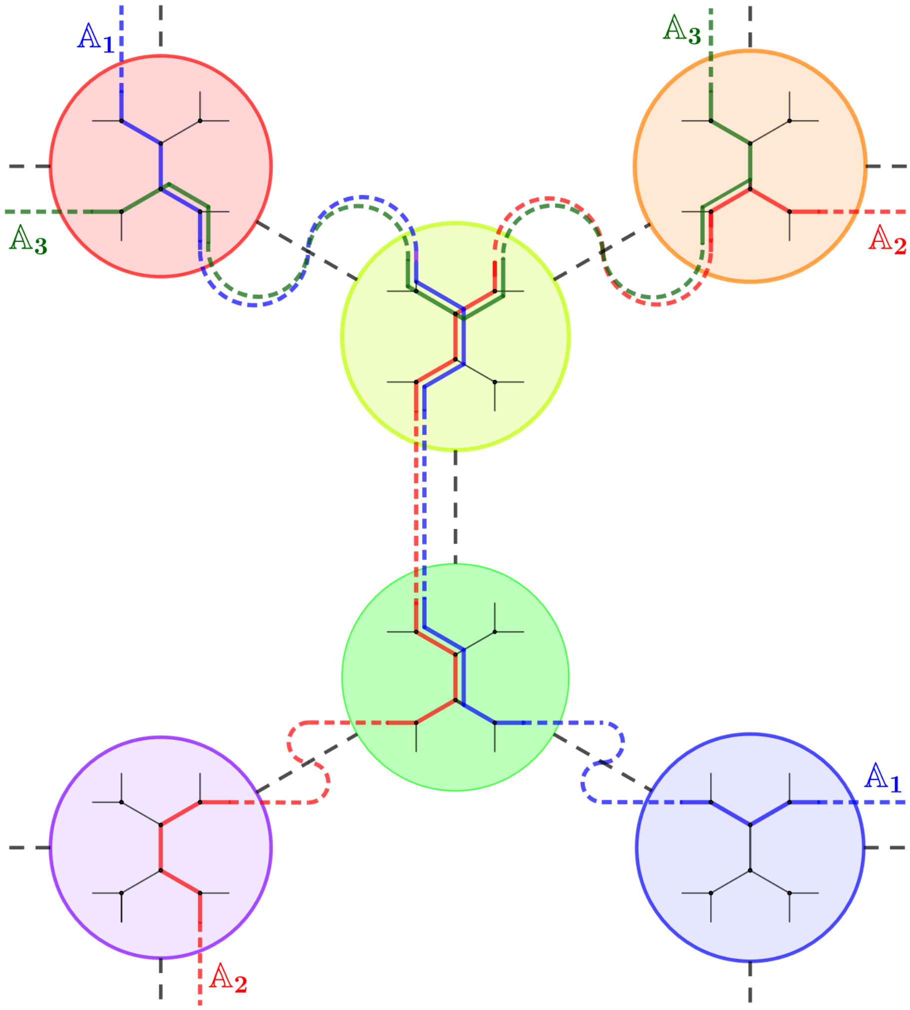

Figure 3: Three apartments in the $\mathbb{Z}^2$-building of SL_2 over a $\mathbb{Z}^2$-valued field.

By construction, any apartment of $\mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \mathbb{K})$ is isomorphic to $\mathbb{Z}^2$. If we fix a basis (e_1, e_2) of V and we denote by $\mathbb{A}$ the apartment associated to this basis, the stabilizer of $\mathbb{A}$ in $\mathrm{SL}(V)$ is the group N of elements of $\mathrm{SL}(V)$ whose matrices in the basis (e_1, e_2) are of the form:

$$\begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}.$$

The group N acts on $\mathbb{A}$ through the quotient N/T_b where T_b is the group of elements of $\mathrm{SL}(V)$ whose matrices in the basis (e_1, e_2) are of the form:

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

with $a \in \mathbb{O}^\times$. The quotient $\widetilde{W} := N/T_b$ is called the **extended affine affine Weyl group** and it is spanned by the elements:

$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, w_1 = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}.$$

One can then check that the set $\{0,1\}^2$ is a fundamental domain for the action of $\widetilde{W}$ on $\mathbb{A}$.

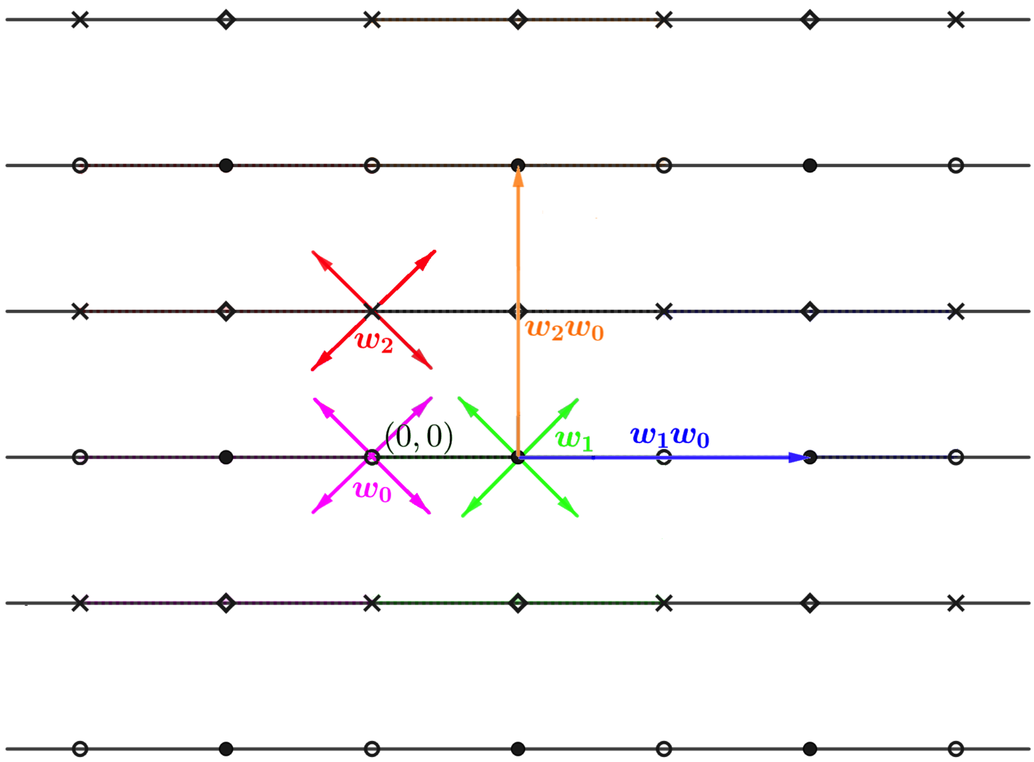

Figure 4: The action of the extended affine Weyl group on the standard apartment of the lattice building of $\mathrm{SL}(V)$ when V is a 2-dimensional vector space over a field $\mathbb{K}$ endowed with a valuation $\omega : \mathbb{K}^\times \to \mathbb{Z}^2$.

2. The totally ordered ring R_n

In the classical case studied by Bruhat and Tits, where one works with reductive groups over a $\mathbb{Z}$-valued field, the apartments of associated buildings are **real** affine spaces. The choice of the real field as the base field stems from the fact that the valuation group $\mathbb{Z}$ can be viewed as an ordered subgroup of $\mathbb{R}$.

In this whole section, we will focus on the case of reductive groups defined over a $\mathbb{Z}^n$-valued field for some n. It seems then natural to replace $\mathbb{R}$ with the totally ordered group $\mathbb{R}^n$, which indeed contains $\mathbb{Z}^n$. However, $\mathbb{R}^n$ is not a field, and hence one cannot directly model apartments using $\mathbb{R}^n$-affine spaces at first glance.

One possible idea is to equip $\mathbb{R}^n$ with a ring structure that is compatible with its structure as an ordered abelian group. This is easily achieved by identifying $\mathbb{R}^n$ with the ring $R_n := \mathbb{R}[[\epsilon]]/(\epsilon^n)$, endowed with the lexicographic order in the $\mathbb{R}$-basis $(1, \epsilon, \epsilon^2, ..., \epsilon^{n-1})$. Therefore, in the following, we will often work with this ring. Hence, it is appropriate to introduce a number of conventions, notations, and associated definitions:

General Notations and Conventions. Throughout, let n be a fixed integer, and denote by R_n the ring $\mathbb{R}[[\epsilon]]/(\epsilon^n)$, equipped with the lexicographic order in the $\mathbb{R}$-basis $(1, \epsilon, \epsilon^2, ..., \epsilon^{n-1})$. For each $\lambda \in R_n$, we denote:

$$R_n^{>\lambda} := \{\mu \in R_n \mid \mu > \lambda\}, \qquad R_n^{\geq\lambda} := \{\mu \in R_n \mid \mu \geq \lambda\}.$$

Affine Geometry over R_n. A principal homogeneous principal space A under a finitely generated R_n-module V is called an **R_n-affine space**. For $a, b \in A$ and $v \in V$, we generally write $a + v$ instead of $v \cdot a$, and denote $b - a$ as the unique element $w \in V$ such that $b = a + w$. If A and A' are two R_n-affine spaces with underlying R_n-modules V and V', a map $f : A \to A'$ is called **affine** if it has the form:

$$f : a \mapsto o' + \overrightarrow{f}(a - o)$$

for some $o \in A$, $o' \in A'$, and $\overrightarrow{f} \in \mathrm{Hom}_{R_n-\mathrm{mod}}(V, V')$. We denote $\mathrm{Aff}_{R_n}(A)$ the group of affine automorphisms of A.

R_n-Distances. Given any set I, a map $d : I \times I \to R_n$ is called an **R_n-distance** if:

(D1) *(Positivity)* For all $x, y \in I$, we have $d(x, y) \geq 0$;

(D2) *(Separation)* For all $x, y \in I$, $d(x, y) = 0$ if and only if $x = y$;

(D3) *(Symmetry)* For all $x, y \in I$, we have $d(x, y) = d(y, x)$;

(D4) *(Triangle inequality)* For all $x, y, z \in I$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

3. Abstract Definition of R_n-Buildings

Let us start by defining the notion of an R_n-building in the sense of Bennett. To do this, we begin by describing the geometry and combinatorics of the corresponding apartments.

a) Affine Apartments over R_n

Let $V_{\mathbb{R}}$ be a finite-dimensional $\mathbb{R}$-vector space and $\Phi \subset (V_{\mathbb{R}})^* := \mathrm{Hom}(V_{\mathbb{R}}, \mathbb{R})$ a **root system** (if you are not familiar with this notion, see [Bou81, Déf. 6.1.1]). We assume Φ spans $(V_{\mathbb{R}})^*$ as an $\mathbb{R}$-vector space, and we denote by $\Phi^\vee \subset V_{\mathbb{R}}$ the dual root system of Φ. There exists a bijection $\vee : \Phi \to \Phi^\vee$ such that $\alpha(\alpha^\vee) = 2$ for all $\alpha \in \Phi$. Each $\alpha \in \Phi$ induces two reflections $r_\alpha \in \mathrm{GL}(V_{\mathbb{R}}^*)$ and $s_\alpha \in \mathrm{GL}(V_{\mathbb{R}})$ such that $r_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha$ and $s_\alpha(\beta^\vee) = \beta^\vee - \alpha(\beta^\vee)\alpha^\vee$ for all $\beta \in \Phi$. The subgroup of $\mathrm{GL}(V_{\mathbb{R}}^*)$ generated by the r_α and the subgroup of $\mathrm{GL}(V_{\mathbb{R}})$ generated by the s_α are canonically isomorphic. Henceforth, we denote them both as W and call them the **vectorial Weyl group**. This is a Coxeter group.

Now consider the R_n-module $V_{R_n} := V_{\mathbb{R}} \otimes R_n$. Let $\mathbb{A}_{R_n}$ be an R_n-affine space with underlying R_n-module V_{R_n}, and fix a point o in $\mathbb{A}_{R_n}$. Each pair $(\alpha, \lambda) \in \Phi \times R_n$ induces an affine map $\theta_{\alpha,\lambda} : \mathbb{A}_{R_n} \to R_n$ such that $\theta_{\alpha,\lambda}(x) = \alpha(x - o) + \lambda$ for all $x \in \mathbb{A}_{R_n}$. This allows us to introduce the "affine hyperplane" $H_{\alpha,\lambda} := \theta_{\alpha,\lambda}^{-1}(\{0\})$, the "open affine half-space" $\mathring{D}_{\alpha,\lambda} := \theta_{\alpha,\lambda}^{-1}(R_n^{>0})$, and the "closed affine half-space" $D_{\alpha,\lambda} := \theta_{\alpha,\lambda}^{-1}(R_n^{\geq 0})$.

Definition 3.1. *An* ***affine apartment*** *over R_n is given by a tuple $\underline{\mathbb{A}_{R_n}} = \left(\mathbb{A}_{R_n}, V_{\mathbb{R}}, \Phi, (\Gamma_\alpha)_{\alpha\in\Phi}, \widehat{W}\right)$ such that:*

1. *$V_{\mathbb{R}}$ is a finite-dimensional $\mathbb{R}$-vector space;*
2. *Φ is a root system in $(V_{\mathbb{R}})^*$;*
3. *$\mathbb{A}_{R_n}$ is an R_n-affine space with underlying R_n-module V_{R_n}, equipped with the topology generated by the $\mathring{D}_{\alpha,\lambda}$;*
4. *$(\Gamma_\alpha)_{\alpha\in\Phi}$ is a family of unbounded subsets of R_n containing 0, satisfying the following property:*

   Let $\mathscr{H} = \{H_{\alpha,\lambda} \mid \alpha \in \Phi, \lambda \in \Gamma_\alpha\}$. For $H = H_{\alpha,\lambda} \in \mathscr{H}$, let r_H be the affine reflection in $\mathbb{A}_{R_n}$ fixing H and whose underlying linear part is given by r_α. Then each r_H stabilizes $\mathscr{H}$.

5. *$\widehat{W}$ is a subgroup of $W \ltimes V_{R_n}$ containing the r_H for $H \in \mathscr{H}$ and stabilizing $\mathscr{H}$.*

The group $\widehat{W}$ is called the **extended affine Weyl group**. Given $\alpha \in \Phi$ and $\lambda \in \Gamma_\alpha$, a set of the form $D_{\alpha,\lambda}$ (resp. $\mathring{D}_{\alpha,\lambda}$, resp. $H_{\alpha,\lambda}$) is called a **half-apartment** (resp. an **open half-apartment**, resp. a **wall**) of $\mathbb{A}_{R_n}$. A subset of $\mathbb{A}_{R_n}$ is said to be **enclosed** if it is a finite intersection of closed half-apartments. A **Weyl chamber** in $\mathbb{A}_{R_n}$ is a set C of the form:

$$\bigcap_{\alpha\in\Delta} D_{\alpha,\lambda_\alpha}$$

where Δ is a basis of the root system Φ and the λ_α are elements in R_n. A **face** of C is a subset of the form:

$$\bigcap_{\alpha\in\Delta'} H_{\alpha,\lambda_\alpha} \cap \bigcap_{\alpha\in\Delta\setminus\Delta'} D_{\alpha,\lambda_\alpha}$$

for some $\Delta' \subset \Delta$. The **vertex** of C is its unique face of dimension 0. It is the only point contained in the intersection $\bigcap_{\alpha\in\Delta} H_{\alpha,\lambda_\alpha}$.

In the sequel, it will be important to endow affine apartments over R_n with an R_n-distance. For that purpose, we fix some subset of positive roots Φ^+ of Φ and we set:

$$\|x\|_{R_n} = \sum_{\alpha \in \Phi^+} |\alpha(x)| \in R_n$$

for $x \in V_{R_n}$. Here, the absolute value of an element $\lambda \in R_n$ is defined as:

$$|\lambda| = \begin{cases} -\lambda & \text{if } \lambda \in R_n \\ \lambda & \text{otherwise.} \end{cases}$$

In this way, we get a W-invariant map $\|\cdot\|_{R_n} : V_{R_n} \to R_n^{\geq 0}$ that does not depend on the choice of Φ^+ and that satisfies:

$$\begin{aligned} \|v + w\|_{R_n} &\leq \|v\|_{R_n} + \|w\|_{R_n}, \\ \|\lambda v\|_{R_n} &= \|v\|_{R_n} |\lambda|, \end{aligned}$$

for any $v, w \in V_{R_n}$ and any $\lambda \in R_n$. Therefore the map $d^{\mathrm{std}}_{R_n} : \mathbb{A}_{R_n} \times \mathbb{A}_{R_n} \to R_n^{\geq 0}$ given by $d^{\mathrm{std}}_{R_n}(x, y) = \|y - x\|_{R_n}$ defines an R_n-distance on $\mathbb{A}_{R_n}$ that is $\widehat{W}$-invariant.

For $x \in \mathbb{A}_{R_n}$ and $\varepsilon \in R_n^{>0}$, we denote by $B_{R_n}(x, \varepsilon)$ the set $\{y \in \mathbb{A}_{R_n} |\ d(x, y) < \varepsilon\}$. The topology of $\mathbb{A}_R$ described in Definition 3.1 thanks to the sets $\mathring{D}_{\alpha,\lambda}$ coincides with the topology that has the $B_{R_n}(x, \varepsilon)$ as a base.

b) Bennett's higher buildings

We are now able to define the notion of an R_n-building. For that purpose, we consider an affine apartment $\underline{\mathcal{A}_{R_n}} = (\mathcal{A}_{R_n}, V_{\mathbb{R}}, \Phi, (\Gamma_\alpha)_{\alpha \in \Phi}, \widehat{W})$ over R_n. We say that a set A is an **apartment of type** $\underline{\mathcal{A}_{R_n}}$ if it is equipped with a non-empty set $\mathrm{Isom}(\mathcal{A}_{R_n}, A)$ of bijections $f : \mathcal{A}_{R_n} \to A$ such that $\mathrm{Isom}(\mathcal{A}_{R_n}, A) = \{f_0 \circ w \mid w \in \widehat{W}\}$ for every $f_0 \in \mathrm{Isom}(\mathcal{A}_{R_n}, A)$. An **isomorphism** between two apartments A and A' is a bijection $\phi : A \to A'$ for which there exists $f_0 \in \mathrm{Isom}(\mathcal{A}_{R_n}, A)$ such that $\phi \circ f_0 \in \mathrm{Isom}(\mathcal{A}_{R_n}, A')$.

Given an apartment A of type $\underline{\mathcal{A}_{R_n}}$, we can choose an element $f \in \mathrm{Isom}(\mathcal{A}_{R_n}, A)$. This allows us to transfer the structures of R_n-affine space and topological space from $\underline{\mathcal{A}_{R_n}}$ to A, and to define half-apartments, walls, enclosures, and Weyl chambers in A. These geometric and combinatorial structures of A do not depend on the choice of f.

Definition 3.2 (Bennett [Ben94]). *Let $\underline{\mathcal{A}_{R_n}}$ be an affine apartment and d be an R_n-distance on $\mathcal{A}_{R_n}$ invariant under the action of the extended affine Weyl group $\widehat{W}$. An R_n-building is a set $\mathcal{I}$ equipped with a covering $\mathscr{A}$ by subsets called apartments such that:*

(A1) Each $A \in \mathscr{A}$ is equipped with an apartment structure of type $\underline{\mathcal{A}_{R_n}}$.

(A2) If A, A' are two apartments, then $A \cap A'$ is an enclosed subset of A and there exists an isomorphism $\phi : A \to A'$ fixing $A \cap A'$.

(A3) For any pair of points in $\mathcal{I}$, there exists an apartment containing both of them.

Axioms (A1)–(A3) then define a map $d_{\mathcal{I}} : \mathcal{I} \times \mathcal{I} \to R_n$ as follows: if x, y are two elements in $\mathcal{I}$, choose an apartment $A \in \mathscr{A}$ containing x and y, and an isomorphism $f \in \mathrm{Isom}(\mathcal{A}_{R_n}, A)$, then define $d_{\mathcal{I}}(x, y) := d(f^{-1}(x), f^{-1}(y))$. This construction is independent of the choices made. With the definitions from Section 2, the map $d_{\mathcal{I}}$ automatically satisfies the axioms of an R_n-distance except for the triangle inequality.

(A4) If C_1 and C_2 are two Weyl chambers in $\mathcal{I}$, there exist two Weyl sub-chambers $C_1' \subseteq C_1$ and $C_2' \subseteq C_2$ that are contained in a common apartment.

(A5) For any apartment A and any $x \in A$, there exists a map $\rho_{A,x} : \mathcal{I} \to A$ such that $\rho_{A,x}|_A = \mathrm{id}_A$, $\rho_{A,x}^{-1}(\{x\}) = \{x\}$, and $d_{\mathcal{I}}(\rho_{A,x}(y), \rho_{A,x}(z)) \leq d_{\mathcal{I}}(y, z)$ for all y and z in $\mathcal{I}$.

(A6) Let A_1, A_2, A_3 be three apartments such that $A_1 \cap A_2$, $A_2 \cap A_3$, and $A_3 \cap A_1$ are half-apartments. Then $A_1 \cap A_2 \cap A_3$ is non-empty.

Axiom (A5) automatically imposes that $d_{\mathcal{I}}$ satisfies the triangle inequality, hence $d_{\mathcal{I}}$ is an R_n-distance.

Remark 3.3. In the case $n = 1$, axioms (A1)-(A5) coincide with axioms (I1)-(I5) of Definition 0.1. Moreover, they imply axiom (A6) (cf. [Parr00, Sec. II.1.4]), so $\mathbb{R}$-buildings in the sense of Bennett are the usual Euclidean buildings.

c) Other useful axiomatizations of higher buildings

In the article [BS14], Bennett and Schwer introduce several other axiomatizations of higher buildings. One of them will be particularly useful in the sequel.

Let $\mathcal{I}$ be a set satisfying axioms (A1), (A2), and (A3), and consider the following additional axioms:

(TI) *(Triangle Inequality)* The map $d_{\mathcal{I}}$ constructed thanks to Axioms (A1)–(A3) satisfies the triangle inequality.

(EC) *(Exchange Condition)* Given two apartments A and B intersecting in a half-apartment M with boundary wall H, the set $(A \setminus B) \cup (B \setminus A) \cup H$ is also an apartment.

(SC) *(Sundial Configuration)* Given an apartment A and a Weyl chamber C such that $P := A \cap C$ is a codimension 1 face of C, if H is the wall of A containing P, then there exist two distinct apartments A_1 and A_2 containing $C \cup H$ and such that $A_1 \cap A$ and $A_2 \cap A$ are both half-apartments in A.

(LA) *(Large Atlas)* If C and C' are two Weyl chambers with respective vertices x and x', then there exist a neighborhood Ω_x of x in C and a neighborhood $\Omega_{x'}$ of x' in C' that are contained in the same apartment.

(GG) *(Locally a Large Atlas)* If C and C' are two Weyl chambers with the same vertex x, then there exist neighborhoods Ω_x and Ω_x' of x in C and C' respectively, contained in the same apartment.

(CO) *(Opposite Chambers)* Let w_0 be the longest element of the vectorial Weyl group W (cf. [Dav07, Sec. 4.6]). Given two Weyl chambers C and C' with the same vertex x, we say they are opposite if there exist neighborhoods Ω_x and Ω'_x of x in C and C' respectively, contained in an apartment A, such that $\Omega'_x - x = w_0 \cdot (\Omega_x - x)$. If C and C' are such chambers, there exists a unique apartment containing both.

(sFC) *(Strong Finite Cover)* For any pair of points x and y, any apartment A containing x and y and any Weyl chamber C with vertex z, the segment:

$$[x, y] := \{\zeta : d_{\mathcal{I}}(x, \zeta) + d_{\mathcal{I}}(\zeta, y)\}$$

is contained in a finite union of Weyl chambers based at z such that each of these Weyl chambers is contained in a common apartment with some open neighborhood of z in C.

(BI) *(Building at Infinity)* Say that two Weyl chambers are parallel if they share a Weyl sub-chamber. The set $\partial\mathcal{I}$ of parallel classes of Weyl chambers in $\mathcal{I}$ is a spherical building with apartments the boundaries ∂A of apartments A in $\mathcal{I}$.

Bennett and Schwer then prove the following result:

Theorem 3.4 (Bennet-Schwer [BS14, Th. 3.3])**.** *Let $\underline{\mathcal{A}_{R_n}}$ be an affine apartment and d be an R_n-distance on $\mathcal{A}_{R_n}$ invariant under the action of the extended affine Weyl group $\widehat{W}$. Let $\mathcal{I}$ be a set equipped with a covering $\mathscr{A}$ by apartments satisfying axioms (A1), (A2) and (A3). The following assertions are equivalent:*

(i) $\mathcal{I}$ is an R_n-building, that is it satisfies axioms (A4), (A5) and (A6).

(ii) $\mathcal{I}$ satisfies axioms (A4), (A5) and (EC).

(iii) $\mathcal{I}$ satisfies axioms (A4), (A5) and (SC).

(iv) $\mathcal{I}$ satisfies axioms (A4), (TI) and (A6).

(v) $\mathcal{I}$ satisfies axioms (A4), (TI) and (SC).

(vi) $\mathcal{I}$ satisfies axioms (A4), (TI) and (EC).

(vii) $\mathcal{I}$ satisfies axioms (TI), (GG) and (CO).

(viii) $\mathcal{I}$ satisfies axioms (GG) and (CO).

(ix) $\mathcal{I}$ satisfies axioms (LA) and (CO).

(x) $\mathcal{I}$ satisfies axioms (A4), (sFC) and (A6).

(xi) $\mathcal{I}$ satisfies axioms (A4), (sFC) and (EC).

(xii) $\mathcal{I}$ satisfies axioms (A4), (sFC) and (SC).

(xiii) $\mathcal{I}$ satisfies axioms (A4), (BI) and (A6).

(xiv) $\mathcal{I}$ satisfies axioms (A4), (BI) and (EC).

(xv) $\mathcal{I}$ satisfies axioms (A4), (BI) and (SC).

The axiomatization of higher buildings given by statement (viii) will be used subsequently.

4. The space $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$

Let $\mathbb{K}$ be a field equipped with a surjective valuation $\omega : \mathbb{K}^\times \to \mathbb{Z}^n$, and let $\mathbf{G}$ be a quasi-split $\mathbb{K}$-reductive group. Denote by $\mathbf{S}$ a maximal split torus in $\mathbf{G}$, and let $\mathbf{T}$ and $\mathbf{N}$ be the centralizer and the normalizer of $\mathbf{S}$ in $\mathbf{G}$, respectively. Let $\Phi = \Phi(\mathbf{G}, \mathbf{S})$ be the root system associated with $(\mathbf{S}, \mathbf{G})$. Note that, since we consider the root system associated to $(\mathbf{G}, \mathbf{S})$ and not the one associated to $(\mathbf{G}, \mathbf{T})$, it might happen that Φ is not reduced: more precisely, it might happen that, for some $\alpha \in \Phi$, the intersection $\mathbb{Z}\alpha \cap \Phi$ is not $\{\pm\alpha\}$ but instead $\{\pm\alpha, \pm 2\alpha\}$.

In this section, we briefly outline how one can associate to the triplet $(\mathbb{K}, \omega, \mathbf{G})$ a space $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ on which the group $\mathbf{G}(\mathbb{K})$ acts naturally. The method is analogous to that used by Bruhat and Tits to construct the building of $\mathbf{G}$ in the case $n = 1$.

Step 1: Definition of a model apartment with an action of the normalizer $\mathbf{N}$. Let $X_*(\mathbf{S})$ (resp. $X^*(\mathbf{S})$) denote the cocharacter module of $\mathbf{S}$ (resp. the character module of $\mathbf{S}$), and consider the $\mathbb{R}$-vector spaces $V_1 := X_*(\mathbf{S}) \otimes \mathbb{R}$ and $V_1^* := X^*(\mathbf{S}) \otimes \mathbb{R}$. The natural pairing $X_*(\mathbf{S}) \otimes X^*(\mathbf{S}) \to \mathbb{Z}$ induces a pairing of $\mathbb{R}$-vector spaces $\langle \cdot, \cdot \rangle : V_1 \otimes V_1^* \to \mathbb{R}$, allowing us to identify V_1 with $\mathrm{Hom}_{\mathbb{R}}(V_1^*, \mathbb{R})$. Moreover, the module $X^*_{\mathbb{K}}(\mathbf{T})$ of characters of $\mathbf{T}$ that are defined over $\mathbb{K}$ is a finite index subgroup of $X^*(\mathbf{S})$ (exercise!). Hence $V_1^* \cong X^*_{\mathbb{K}}(\mathbf{T}) \otimes \mathbb{R}$, and:

$$V_1 \cong \mathrm{Hom}_{\mathbb{R}}(X^*_{\mathbb{K}}(\mathbf{T}) \otimes \mathbb{R}, \mathbb{R}).$$

By change of basis, we obtain:

$$V_1 \otimes R_n \cong \mathrm{Hom}_{\mathbb{R}}(X^*_{\mathbb{K}}(\mathbf{T}) \otimes \mathbb{R}, R_n).$$

We can therefore define a morphism:

$$\begin{aligned} \rho : \mathbf{T}(\mathbb{K}) &\to V_1 \otimes R_n \\ t &\mapsto \rho(t) \end{aligned}$$

where:

$$\begin{aligned} \rho(t) : X^*_{\mathbb{K}}(\mathbf{T}) \otimes \mathbb{R} &\to R_n \\ \chi \otimes \lambda &\mapsto -\lambda\omega(\chi(t)). \end{aligned}$$

Introduce now the $\mathbb{R}$-vector space:

$$V_{\mathbb{R}} := V_1/\Phi^\perp$$

where

$$\Phi^\perp := \{v \in X_*(\mathbf{S}) \otimes \mathbb{R} \,|\, \forall \alpha \in \Phi, \alpha(v) = 0\}.$$

Set $V_{R_n} := V_{\mathbb{R}} \otimes R_n$ and consider an R_n-affine space $\mathbb{A}_{R_n}$ with underlying R_n-module V_{R_n}. The morphism ρ then induces an action of $\mathbf{T}(\mathbb{K})$ on V_{R_n} by translations, and the Weyl group W associated to the root system Φ acts on V_{R_n} by base change. By identifying $\mathbb{A}_{R_n}$ and V_{R_n} via the choice of a suitable origin $o \in \mathbb{A}_{R_n}$, we obtain actions of $\mathbf{T}(\mathbb{K})$ and $W = \mathbf{N}(\mathbb{K})/\mathbf{T}(\mathbb{K})$ on $\mathbb{A}_{R_n}$. These can actually be combined to obtain a unique action of $\mathbf{N}(\mathbb{K})$ on $\mathbb{A}_{R_n}$ by R_n-affine transformations. In other words, there exists a group morphism $\nu : \mathbf{N}(\mathbb{K}) \to \mathrm{Aff}_{R_n}(\mathbb{A}_{R_n})$ making the diagram commute:

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{T}(\mathbb{K}) \arrow[r] \arrow[d, "\rho_0"] & \mathbf{N}(\mathbb{K}) \arrow[r] \arrow[d, "\nu"] & W \arrow[r] \arrow[d, "j"] & 1 \\ 1 \arrow[r] & V_{R_n} \arrow[r] & \mathrm{Aff}_{R_n}(\mathbb{A}_{R_n}) \arrow[r] & \mathrm{GL}_{R_n}(V_{R_n}) \arrow[r] & 1, \end{tikzcd}$$

where ρ_0 is the composition of $\rho : \mathbf{T}(\mathbb{K}) \to V_1 \otimes R_n$ followed by the projection $V_1 \otimes R_n \to V_{R_n}$ and j is the natural inclusion $W \subset \mathrm{GL}(V_{\mathbb{R}}) \subset \mathrm{GL}_{R_n}(V_{R_n})$.

Sketch of proof. Let W' be the push-out of the morphism $\rho_0 : \mathbf{T}(\mathbb{K}) \to V_{R_n}$ and the inclusion $\mathbf{T}(\mathbb{K}) \subseteq \mathbf{N}(\mathbb{K})$. The group W' is then an extension of W by V:

$$1 \to V_{R_n} \to W' \to W \to 1. \tag{1}$$

The group W is finite, while the group V_{R_n} is a $\mathbb{Q}$-vector space. A general argument in abstract group theory then shows that exact sequence (1) splits, so that $W' = V_{R_n} \rtimes W$. Hence j induces a morphism:

$$j' : W' = V \rtimes W \to V \rtimes \mathrm{GL}(V_{R_n}) = \mathrm{Aff}_{R_n}(V_{R_n}).$$

By composing the natural map $\mathbf{N}(\mathbb{K}) \to W'$ and j', we get the desired morphism:

$$\nu : \mathbf{N}(\mathbb{K}) \to \mathrm{Aff}_{R_n}(V_{R_n}).$$

□

The extended affine Weyl group $\widehat{W}$ is defined as the image of ν.

Step 2: Definition of parahoric subgroups. For any root $\alpha \in \Phi$, there exists a unique $\mathbb{K}$-subgroup of $\mathbf{G}$, denoted by $\mathbf{U}_\alpha$, which is closed, connected, unipotent, normalized by $\mathbf{T}$ and whose Lie algebra is $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$. It is called the *root group* of $\mathbf{G}$ with respect to α. We let $\pi : \mathbf{G}^\alpha \to \langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} \rangle$ be the universal cover of $\langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} \rangle$.

When $\alpha/2, 2\alpha \notin \Phi$, one can find a finite extension $\mathbb{L}_\alpha$ of $\mathbb{K}$ and an isomorphism $\xi_\alpha : R_{\mathbb{L}_\alpha/\mathbb{K}}(\mathrm{SL}_{2,\mathbb{L}_\alpha}) \to \mathbf{G}^\alpha$. But the root groups of $\mathrm{SL}_{2,\mathbb{L}_\alpha}$ associated to the maximal torus given by diagonal matrices can be parametrized as follows:

$$\begin{array}{rccc} y_- : & \mathbb{G}_{a,\mathbb{L}_\alpha} & \to & \mathrm{SL}_{2,\mathbb{L}_\alpha} \\ & v & \mapsto & \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \end{array} \qquad \begin{array}{rccc} y_+ : & \mathbb{G}_{a,\mathbb{L}_\alpha} & \to & \mathrm{SL}_{2,\mathbb{L}_\alpha} \\ & u & \mapsto & \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \end{array}$$

One can then get isomorphims:

$$x_{\pm\alpha} := \pi \circ \xi_\alpha \circ R_{\mathbb{L}_\alpha/\mathbb{K}}(y_\pm) : R_{\mathbb{L}_\alpha/\mathbb{K}}(\mathbb{G}_\mathrm{a}) \to \mathbf{U}_{\pm\alpha}.$$

By composing x_α^{-1} with the (unique) extension $\omega_{\mathbb{L}_\alpha} : \mathbb{L}_\alpha \to R_n \cup \{\infty\}$ of ω to $\mathbb{L}_\alpha$, we can define a "valuation":

$$\begin{aligned} \varphi_\alpha : \mathbf{U}_\alpha(\mathbb{K}) &\to R_n \\ x_\alpha(u) &\mapsto \omega_{\mathbb{L}_\alpha}(u). \end{aligned}$$

We define the value group $\Gamma_\alpha := \varphi_\alpha(\mathbf{U}_\alpha(\mathbb{K}) \setminus \{1\})$.

When $2\alpha \in \Phi$, one can find a tower of finite extensions $\mathbb{L}_\alpha/\mathbb{L}'_\alpha/\mathbb{K}$ with $[\mathbb{L}_\alpha : \mathbb{L}'_\alpha] = 2$ such that, if τ stands for the non-trivial element in $\mathrm{Gal}(\mathbb{L}_\alpha/\mathbb{L}'_\alpha)$ and h is the hermitian form on $\mathbb{L}_\alpha^3$ given by:

$$h : (x_{-1}, x_0, x_1) \mapsto \sum_{i=-1}^{1} x_i {}^\tau x_{-i},$$

then there is an isomorphism $\xi_\alpha : R_{\mathbb{L}'_\alpha/\mathbb{K}}(\mathrm{SU}(h)) \to \mathbf{G}^\alpha$. But the root groups of $\mathrm{SU}(h)$ associated to the maximal torus given by diagonal matrices can be parametrized as follows:

$$\begin{array}{rccc} y_- : & \mathbf{H}_\alpha & \to & \mathrm{SU}(h) \\ & (u,v) & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ -v & -{}^\tau u & 1 \end{pmatrix} \end{array} \qquad \begin{array}{rccc} y_+ : & \mathbf{H}_\alpha & \to & \mathrm{SU}(h) \\ & (u,v) & \mapsto & \begin{pmatrix} 1 & -{}^\tau u & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

where $\mathbf{H}_\alpha$ is the $\mathbb{L}'_\alpha$-subvariety of $R_{\mathbb{L}_\alpha/\mathbb{L}'_\alpha}(\mathbb{A}^2_{\mathbb{L}_\alpha})$ defined by:

$$u {}^\tau u = v + {}^\tau v$$

together with the group law:

$$(u, v), (u', v') \mapsto (u + u', v + v' + {}^\tau u u').$$

One can then get isomorphims:

$$x_{\pm\alpha} := \pi \circ \xi_\alpha \circ R_{\mathbb{L}_\alpha/\mathbb{K}}(y_\pm) : R_{\mathbb{L}'_\alpha/\mathbb{K}}(\mathbf{H}_\alpha) \to \mathbf{U}_{\pm\alpha}.$$

By composing x_α^{-1} with the (unique) extension $\omega_{\mathbb{L}_\alpha} : \mathbb{L}_\alpha \to R_n \cup \{\infty\}$ of ω to $\mathbb{L}_\alpha$, we can define "valuations":

$$\begin{aligned} \varphi_\alpha : \mathbf{U}_\alpha(\mathbb{K}) &\to R_n \\ x_\alpha(u,v) &\mapsto \frac{1}{2}\omega_{\mathbb{L}_\alpha}(v), \end{aligned} \qquad \begin{aligned} \varphi_{2\alpha} : \mathbf{U}_\alpha(\mathbb{K}) &\to R_n \\ x_\alpha(0,v) &\mapsto \omega_{\mathbb{L}_\alpha}(v). \end{aligned}$$

We define the value groups:

$$\begin{aligned} \Gamma_\alpha &:= \varphi_\alpha(\mathbf{U}_\alpha(\mathbb{K}) \setminus \{1\}), \\ \Gamma'_\alpha &:= \Big\{\varphi_\alpha(x) : \ x \in \mathbf{U}_\alpha(\mathbb{K}) \setminus \{1\} \text{ and } \forall y \in \mathbf{U}_{2\alpha}(\mathbb{K}), \varphi_\alpha(xy) \le \varphi_\alpha(x)\Big\}, \\ \Gamma_{2\alpha} &:= \varphi_{2\alpha}(\mathbf{U}_{2\alpha}(\mathbb{K}) \setminus \{1\}). \end{aligned}$$

The valuations φ_α we have just constructed induce filtrations of the root groups $\mathbf{U}_\alpha(\mathbb{K})$. Indeed, for each $\alpha \in \Phi$ and each $\lambda \in R_n$, we introduce the group:

$$U_{\alpha,\lambda} := \{1\} \cup \varphi_\alpha^{-1}(R_n^{\ge\lambda}) \subseteq \mathbf{U}_\alpha(\mathbb{K}).$$

This allows to define the so-called parahoric subgroups. More precisely, for each $x \in \mathbb{A}_{R_n}$, we introduce the groups:

$$\begin{aligned} N_x &:= \mathrm{Stab}_{\mathbf{N}(\mathbb{K})}(x), \\ U_x &:= \langle U_{\alpha,-\alpha(x-o)} \,|\, \alpha \in \Phi \rangle, \\ P_x &:= U_x N_x, \end{aligned}$$

where o is the origin we have chosen in $\mathbb{A}_{R_n}$.

Step 3: Definition of $\mathcal{I}(\mathbb{K},\omega,\mathbf{G})$. We now have all the necessary tools to define $\mathcal{I}(\mathbb{K},\omega,\mathbf{G})$:

Definition 4.1. *The space $\mathcal{I}(\mathbb{K},\omega,\mathbf{G})$ is defined as the quotient of $\mathbf{G}(\mathbb{K})\times\mathbb{A}_{R_n}$ by the following equivalence relation:*

$$(g,x)\sim(h,y)\Leftrightarrow \exists n\in\mathbf{N}(\mathbb{K}),\begin{cases} y=\nu(n)(x), \\ g^{-1}hn\in P_x. \end{cases}$$

The group $\mathbf{G}(\mathbb{K})$ acts on $\mathcal{I}(\mathbb{K},\omega,\mathbf{G})$ by:

$$g\cdot(h,x):=(gh,x),$$

where $g,h\in\mathbf{G}(\mathbb{K})$ and $x\in\mathbb{A}_{R_n}$.

An apartment in $\mathcal{I}(\mathbb{K},\omega,\mathbf{G})$ will then be a subset of the form $\{g\}\times\mathbb{A}_{R_n}$ for some $g\in\mathbf{G}(\mathbb{K})$.

5. Statement of the Main Theorem

We are finally ready to state the Main Theorem from [HIL20].

Theorem 5.1 (Hébert-I.-Loisel, [HIL20, Th. 3.26])**.** *Let us consider:*

an integer $n\geq 1$,

a henselian field $\mathbb{K}$ equipped with a surjective valuation $\omega:\mathbb{K}\to\mathbb{Z}^n\cup\{\infty\}$,

a quasi-split $\mathbb{K}$-reductive (connected) group $\mathbf{G}$,

and introduce the notations of Section 4.1:

$\mathbf{S}$ *a maximal split torus of $\mathbf{G}$,*

Φ *the root system associated with $(\mathbf{G},\mathbf{S})$,*

$X_*(\mathbf{S})$ *the module of cocharacters of $\mathbf{S}$,*

$V_{\mathbb{R}}$ *the quotient of $X_*(\mathbf{S})\otimes_{\mathbb{Z}}\mathbb{R}$ by the orthogonal of Φ,*

$\mathbb{A}_{R_n}$ *an R_n-affine space with underlying R_n-module $V_{\mathbb{R}}\otimes R_n$,*

$(\Gamma_\alpha)_{\alpha\in\Phi}$ *the value groups,*

$\widehat{W}$ *the extended affine Weyl group,*

the space $\mathcal{I}(\mathbb{K},\omega,\mathbf{G})$.

Endow $\mathbb{A}_{R_n}$ with an R_n-distance d that is invariant under the action of the affine Weyl group $\widehat{W}$.

(i) The set $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ is an R_n-building whose apartments are of type:

$$\underline{\mathbb{A}_{R_n}} = (\mathbb{A}_{R_n}, V_{\mathbb{R}}, \Phi, (\Gamma_\alpha)_{\alpha\in\Phi}, \widehat{W}).$$

(ii) Let $d_{\mathcal{I}(\mathbb{K},\omega,\mathbf{G})}$ denote the distance on $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ provided by axioms (A1), (A2) and (A3). The group $\mathbf{G}(\mathbb{K})$ acts on $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ by isometries, and the restriction of this action to apartments is given by R_n-affine transformations. The induced action on the set of apartments is transitive.

(iii) Fix $s \in \{1, \dots, n\}$, introduce the natural projection $\pi^{(0)}_{\leq s} : R_n \to R_s$ and consider the valuation $\omega_{\leq s} = \pi^{(0)}_{\leq s} \circ \omega$. The projection $\pi^{(0)}_{\leq s}$ induces a surjective map:

$$\pi_{\leq s} : \mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) \to \mathcal{I}(\mathbb{K}, \omega_{\leq s}, \mathbf{G}),$$

compatible with the action of the group $\mathbf{G}(\mathbb{K})$. If $\Gamma_{\alpha,\leq s}$ and $\Gamma'_{\alpha,\leq s}$ for $\alpha \in \Phi$ stand for the value groups associated to the construction of the building $\mathcal{I}(\mathbb{K}, \omega_{\leq s}, \mathbf{G})$, then for each point $X = [g, x] \in \mathcal{I}(\mathbb{K}, \omega_{\leq s}, \mathbf{G})$, the fiber $\pi^{-1}_{\leq s}(X)$ is a product:

$$\mathcal{I}_X \times \left(\langle\Phi_x\rangle^{\perp} \otimes_{\mathbb{R}} R_{n-s}\right),$$

where

$$\Phi_x = \begin{cases} \{\alpha \in \Phi \mid -\alpha(x) \in \Gamma_{\alpha,\leq s}\} & \text{if } 2\alpha \notin \Phi \\ \{\alpha \in \Phi \mid -\alpha(x) \in \Gamma'_{\alpha,\leq s}\} & \text{otherwise,} \end{cases}$$

$\langle\Phi_x\rangle^{\perp}$ is the orthogonal of Φ_x in $V_{\mathbb{R}}$, and $\mathcal{I}_X$ is an R_{n-s}-building whose apartments are R_{n-s}-affine spaces with underlying R_{n-s}-module $\left(V_{\mathbb{R}}/\langle\Phi_x\rangle^{\perp}\right) \otimes_{\mathbb{R}} R_{n-s}$.

Remark 5.2. This theorem can be extended to the case where $\mathbb{K}$ is equipped with a valuation taking values in any totally ordered abelian group Λ. To do this, one must introduce the rank $\mathrm{rk}(\Lambda)$ of Λ, which is the totally ordered set of archimedean equivalence classes of Λ, and replace the ring R_n with the sub-$\mathbb{R}$-vector space $\mathfrak{R}^{\mathrm{rk}(\Lambda)}$ of $\mathbb{R}^{\mathrm{rk}(\Lambda)}$ consisting of families $(x_s)_{s\in\mathrm{rk}(\Lambda)}$ with well-ordered[1] support. By endowing this group with the lexicographic order, the so-called Hahn Embedding Theorem then ensures that Λ embeds as an ordered subgroup into $\mathfrak{R}^{\mathrm{rk}(\Lambda)}$. However, unlike R_n, the space $\mathfrak{R}^{\mathrm{rk}(\Lambda)}$ generally lacks a natural ring structure, so the notions of affine geometry from Section 2 must be redefined in this more general context.

The example of SL_2**.** In order to illustrate the previous Theorem, we come back to the example of $\mathrm{SL}(V)$ for V a 2-dimensional vector space over a $\mathbb{Z}^2$-valued field $\mathbb{K}$. We keep all the notations of Section 1.2. According to Theorem 5.1, there exists a fibration

$$\pi := \pi_{\leq 1} : \mathcal{I}(\mathbb{K}, \omega, \mathrm{SL}(V)) \to \mathcal{I}(\mathbb{K}, \omega_1, \mathrm{SL}(V))$$

satisfying the following properties:

(i) $\mathcal{I}(\mathbb{K}, \omega_1, \mathrm{SL}(V))$ is the classical Bruhat-Tits building for $\mathrm{SL}(V)$ over $(\mathbb{K}, \omega_1)$; in particular, its vertices correspond to $\mathcal{O}$-lattices in V modulo homothety;

[1] A totally ordered set is said to be well-ordered if every non-empty subset has a smallest element.

(ii) for each point X lying inside an edge of $\mathcal{I}(\mathbb{K}, \omega_1, \mathrm{SL}(V))$, the fiber $\pi^{-1}(X)$ is isomorphic to the real line $\mathbb{R}$;

(iii) for each vertex X of $\mathcal{I}(\mathbb{K}, \omega_1, \mathrm{SL}(V))$ corresponding to a $\mathcal{O}$-lattice L in V, the fiber $\pi^{-1}(X)$ is isomorphic to the Bruhat-Tits building of $\mathrm{SL}(L/\mathcal{M}L)$ over $(\mathcal{K}_1, \omega_0)$, and its stabilizer in $\mathrm{SL}(V)$ is $\mathrm{SL}(L)$; the latter acts on the fiber via the quotient $\mathrm{SL}(L) \to \mathrm{SL}(L/\mathcal{M}L)$;

(iv) the apartments of $\mathcal{I}(\mathbb{K}, \omega, \mathrm{SL}(V))$ are the inverse images of apartments in $\mathcal{I}(\mathbb{K}, \omega_1, \mathrm{SL}(V))$.

The following figures illustrate these phenomena:

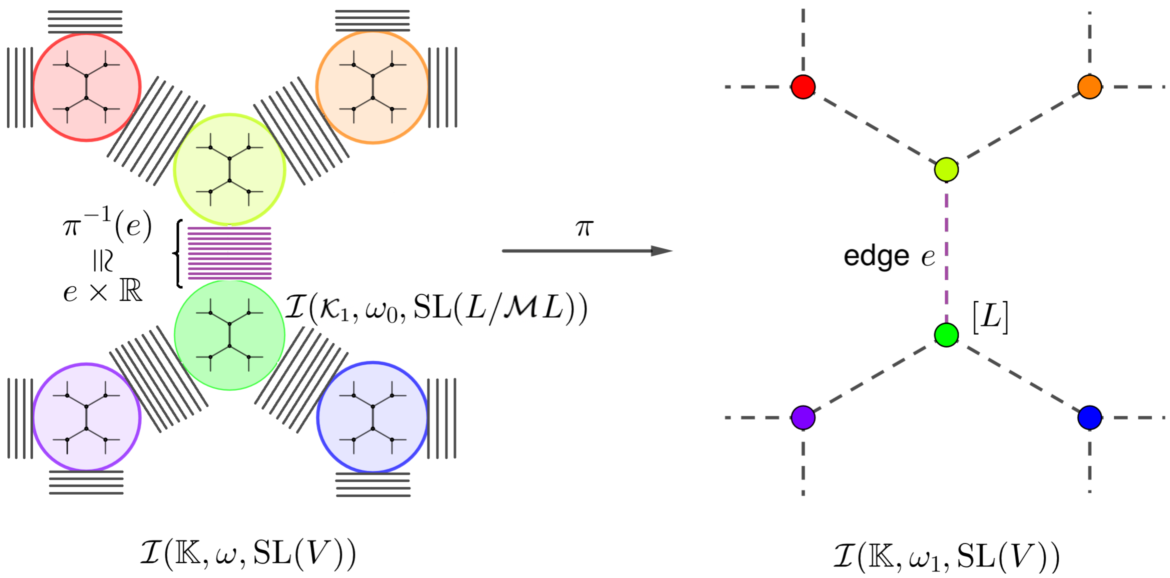

Figure 5: The building of $\mathrm{SL}(V)$ when V is a 2-dimensional vector space over a field $\mathbb{K}$ with a valuation $\omega : \mathbb{K}^\times \to \mathbb{Z}^2$.

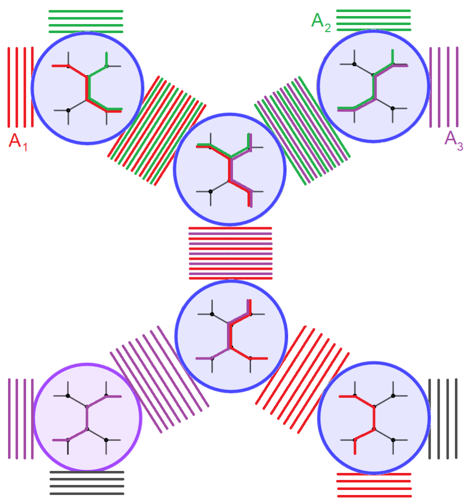

Figure 6: Three apartments A_1 (in red), A_2 (in green), and A_3 (in violet) of the building of $\mathrm{SL}(V)$ when V is a 2-dimensional vector space over a field $\mathbb{K}$ with a valuation $\omega : \mathbb{K}^\times \to \mathbb{Z}^2$.

Relation to the Lattice Building. One can construct an isometric injection $\mathrm{SL}(V)$-invariant $i : \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega) \to \mathcal{I}(\mathbb{K}, \omega, \mathrm{SL}(V))$ that identifies $\mathcal{I}^{\mathcal{L}}(\mathbb{K}, \mathrm{SL}(V), \omega)$ with the vertices of $\mathcal{I}(\mathbb{K}, \omega, \mathrm{SL}(V))$ and induces a bijection:

$$i^* : \{\text{Apartments of } \mathcal{I}(\mathbb{K}, \omega, \mathrm{SL}(V))\} \to \{\text{Apartments of } \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega)\}$$
$$A \mapsto i^{-1}(A).$$

Furthermore, we have a commutative diagram:

$$\begin{tikzcd} \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega) \arrow[r, "i"] \arrow[d, "\pi^{\mathcal{L}}"] & \mathcal{I}(\mathbb{K}, \omega, \mathrm{SL}(V)) \arrow[d, "\pi"] \\ \mathcal{I}^{\mathcal{L}}(\mathrm{SL}(V), \omega_1) \arrow[r, "i_1"] & \mathcal{I}(\mathbb{K}, \omega_1, \mathrm{SL}(V)). \end{tikzcd}$$

Action on an apartment. Let A be an apartment in $\mathcal{I}(\mathbb{K}, \omega, \mathrm{SL}(V))$. Identifying A with $R_2 = \mathbb{R}[[\epsilon]]/(\epsilon^2)$, one can verify that the set

$$\{a + b\epsilon \,|\, 0 \leq a, b \leq 1\} \cup \{a + b\epsilon \,|\, 1 < a < 2,\, 0 < b \leq 1\} \subset R_2$$

is a fundamental domain for the action of the extended affine Weyl group $\widetilde{W}$.

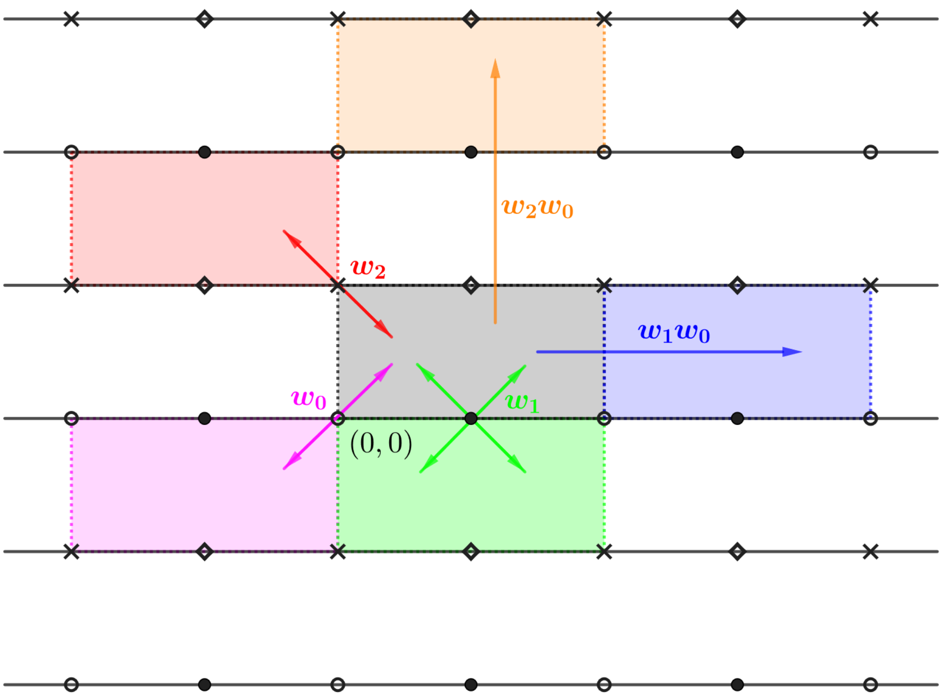

Figure 7: Action of $\widetilde{W}$ on an apartment.

6. General outline of the proof of Theorem 5.1

The proof of Theorem 5.1 is long and technical. This is partly due to the fact that the building $\mathcal{I}_X$ appearing in part (iii) is generally not associated with a $\mathbb{K}$-reductive group. For this reason, we introduce a general and abstract formalism of valued R_n-root data that generalizes the valued root data introduced by Bruhat and Tits in [BT72, Sec. 6.1 & 6.2]. The proof then consists in showing, on the one hand, that to each valued R_n-root data we can associate an R_n-building - which we do by proving axioms (A1), (A2), (A3), (GG) and (CO) as authorized by Theorem 3.4 - and on the other hand, that the spaces $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ and $\mathcal{I}_X$ appearing in Theorem 5.1 are precisely buildings associated to certain valued R_n-root data.

More precisely, here is a more detailed step-by-step proof outline.

Step A. Given an abstract group G and a root system Φ, we define a notion of *R_n-valued root data* for G of type Φ by a system of axioms that are listed in the Appendix. This provides subgroups T, N and $(U_\alpha)_{\alpha\in\Phi}$ within G, as well as "valuations" $(\varphi_\alpha : U_\alpha \to R_n \cup \{\infty\})_{\alpha\in\Phi}$ playing roles analogous to the groups $\mathbf{T}(\mathbb{K})$, $\mathbf{N}(\mathbb{K})$ and $(\mathbf{U}_\alpha(\mathbb{K}))_{\alpha\in\Phi}$ and to the valuations $(\varphi_\alpha)_{\alpha\in\Phi}$ associated with a quasi-split reductive group $\mathbf{G}$ in Section 4.

Step B. Given an R_n-valued root data, we define a notion of *compatible action ν* of N on an R_n-affine space $\mathbb{A}$ by a system of axioms that are listed in the Appendix. This action plays the role of the action of $\mathbf{N}(\mathbb{K})$ on $\mathbb{A}_{R_n}$ associated to a quasi-split reductive group $\mathbf{G}$ in Section 4.

Step C. We study the properties of groups associated to an R_n-valued root datum endowed with a compatible action on an R_n-affine space $\mathbb{A}$. More precisely, following similar constructions to the ones described in Section 4, we introduce for each $x \in \mathbb{A}$ some subgroups N_x, U_x and $P_x = U_x N_x$ of G, and we prove an *Iwasawa decomposition* and a *Bruhat decomposition* in this context. This latter result generalizes the classical Bruhat decomposition for $\mathbb{Z}$-valued fields and a Theorem of Kapranov for split reductive groups over $\mathbb{Z}^2$-valued fields ([Kap01, Prop. (1.2.3)]). The following theorem presents what the Bruhat decomposition says about the groups associated to a reductive group $\mathbf{G}$ in Section 4.

Theorem 6.1 (Bruhat decomposition, [HIL20, Th. 5.37])**.** *We take all the notations of Section 4. Let C, C' be two Weyl chambers of $\mathbb{A}_{R_n}$ with respective vertices $x, x' \in \mathbb{A}_{R_n}$. Let Ω_x and $\Omega_{x'}$ two open neighbourhoods of x and x' in C and C' respectively, and set:*

$$P_{\Omega_x} := \bigcap_{y\in\Omega_x} P_y, \qquad P_{\Omega_{x'}} := \bigcap_{y'\in\Omega_{x'}} P_{y'}.$$

If Ω_x and $\Omega_{x'}$ are sufficiently small, then

$$\mathbf{G}(\mathbb{K}) = P_{\Omega_x}\mathbf{N}(\mathbb{K})P_{\Omega_{x'}}$$

and there is a natural one-to-one correspondence

$$\widehat{W} \to P_{\Omega_x}\backslash\mathbf{G}(\mathbb{K})/P_{\Omega_{x'}}.$$

Step D. Given an R^n-valued root datum D of a group G with a compatible action on an R_n-affine space $\mathbb{A}$, we construct the quotient $\mathcal{I}(D) := (G \times \mathbb{A})/\sim$ with:

$$(g, x) \sim (h, y) \Leftrightarrow \exists n \in N, \begin{cases} y = \nu(n)(x), \\ g^{-1}hn \in U_x. \end{cases}$$

This quotient plays the role here of the space $\mathcal{I}(\mathbb{K},\omega,\mathbf{G})$ associated with a quasi-split reductive group $\mathbf{G}$ in Section 4. It satisfies axiom (A1) by definition and is endowed with an action of G whose restrictions to apartments are affine transformations. By definition, the induced action on the set of apartments is transitive. One can also easily check that, for each $x \in \mathbb{A}$, the stabilizer of x in G is P_x. Indeed,

$$[1,x] = g \cdot [1,x] \iff \exists n \in N, \begin{cases} x = \nu(n^{-1})(x) \\ gn^{-1} \in U_x \end{cases} \iff \exists n \in N_x,\ g \in U_x n$$

since N_x is, by definition, the stabilizer of x in N. Thus, the stabilizer of x in G is $U_x N_x$, which is P_x.

Step E. We use the results from Step C to show that $\mathcal{I}(D)$ satisfies axioms (A2), (A3), (A4), and (GG). For instance:

Proposition 6.2. *We take all the notations of Section 4. Then $\mathcal{I}(\mathbb{K},\omega,\mathbf{G})$ satisfies axiom (LA). In particular, it satisfies axioms (A3) and (GG).*

Proof. Let C and C' be two Weyl chambers with respective vertices x and x'. Since $\mathbf{G}(\mathbb{K})$ acts transitively on the set of apartments, we may assume that $C \subset \mathbb{A}_{R_n}$. Let $g \in \mathbf{G}(\mathbb{K})$ be such that $g^{-1}.C' \subset \mathbb{A}_{R_n}$. By the Bruhat decomposition 6.1, we can find Ω_x and $\Omega_{x'}$ two sufficiently small open neighbourhoods of x and x' in C and C' respectively and write $g = bnb'$ with $b \in P_{\Omega_x}$, $b' \in P_{g^{-1}\Omega_{x'}}$ and $n \in N$. Then $b\Omega_x = \Omega_x \subset b\mathbb{A}_{R_n}$ and

$$\Omega_{x'} = gg^{-1}\Omega_{x'} = bnb'g^{-1}\Omega_{x'} = bng^{-1}\Omega_{x'} \subset b.\mathbb{A}_{R_n}.$$

□

Step F. We consider a quasi-split reductive group $\mathbf{G}$ over a henselian $\mathbb{Z}^n$-valued field $\mathbb{K}$, and follow the constructions explained in Section 4 to associate to it an R_n-valued root data D with a compatible action on an R_n-affine space $\mathbb{A}_{R_n}$. This allows us to identify the space $\mathcal{I}(\mathbb{K},\omega,\mathbf{G})$ with $\mathcal{I}(D)$.

Step G. We construct the fibration $\pi_{\leq s}$ from part (iii). To do so, observe that all the work that was done in the previous steps for the quasi-split reductive group $\mathbf{G}$ over the valued field $(\mathbb{K},\omega)$ can be done over the valued field $(\mathbb{K},\omega_{\leq s})$. In particular, the root group datum associated to $\mathbf{G}$ can be endowed with two valuations and two compatible actions:

- an R_n-valuation $(\varphi_\alpha)_{\alpha\in\Phi}$ induced by ω and a compatible action ν of $\mathbf{N}(\mathbb{K})$ on an R_n-affine space $\mathbb{A}_{R_n}$;
- an R_s-valuation $(\varphi_{\leq s,\alpha})_{\alpha\in\Phi}$ induced by $\omega_{\leq s}$ and a compatible action $\nu_{\leq s}$ of $\mathbf{N}(\mathbb{K})$ on an R_s-affine space $\mathbb{A}_{R_s}$.

The first valuation gives rise to the R_n-valued root group datum D introduced in Step F, and the second to an R_s-valued root group datum that we denote by $D_{\leq s}$.

Now recall that there is a real vector space $V_{\mathbb{R}}$ such that $\mathbb{A}_{R_n}$ is an R_n-affine space with underlying R_n-module $V \otimes R_n$ and $\mathbb{A}_{R_s}$ is an R_s-affine space with underlying R_s-module $V \otimes R_s$. Hence, if we fix two origins $o_n \in \mathbb{A}_{R_n}$ and $o_s \in \mathbb{A}_{R_s}$, the projection

$\pi^{(0)}_{\leq s} : R_n \to R_s$ induces a projection map:

$$\begin{aligned}\pi_{\mathbb{A},\leq s} &: \mathbb{A}_{R_n} \to \mathbb{A}_{R_s}\\ x = o_n + v &\mapsto \pi_{\mathbb{A},\leq s}(x) = o_s + (\mathrm{id}_V \otimes \pi^{(0)}_{\leq s})(v).\end{aligned}$$

This projection map itself naturally induces a surjective map:

$$\pi_{\leq s} : \mathcal{I}(D) \to \mathcal{I}(D_{\leq s})$$

which is compatible with the $\mathbf{G}(\mathbb{K})$-action.

The main result we prove concerning this projection map can be stated as follows:

Theorem 6.3 ([HIL20, Th. 8.15])**.** *For each point $X = [g, x] \in \mathcal{I}(\mathbb{K}, \omega_{\leq s}, \mathbf{G})$, the fiber $\pi^{-1}_{\leq s}(X)$ is a product:*

$$\mathcal{I}(D_X) \times \left(\langle \Phi_x \rangle^{\perp} \otimes_{\mathbb{R}} R_{n-s}\right),$$

for some R_{n-s}-valued root group datum D_X endowed with a compatible action on an R_{n-s}-affine space with underlying R_{n-s}-module $\left(V_{\mathbb{R}}/\langle \Phi_x \rangle^{\perp}\right) \otimes_{\mathbb{R}} R_{n-s}$.

Thanks to Steps D and E, we know that $\mathcal{I}(D_X)$ satisfies axioms (A1), (A2), (A3), (A4), and (GG) with n replaced by $n - s$.

Step H. We prove axiom (CO) both for $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) = \mathcal{I}(D)$ and for the $\mathcal{I}_X = \mathcal{I}(D_X)$ appearing in the fibers of $\pi_{\leq s}$. Projections of Step G play a crucial role in this Step in order to reduce to the case $n = 1$. More precisely, let us set $\mathcal{I} := \mathcal{I}(D) = \mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$. We proceed in four substeps to prove that $\mathcal{I}$ satisfies axiom (CO):

(i) Given a set satisfying axioms (A1)-(A4) and (GG) with $n = 1$, we provide sufficient conditions for it to also satisfy axiom (CO) and hence to be an $\mathbb{R}$-building.

(ii) Take any $s_0 \in \{1, \dots, n\}$. According to Step G, we have three projection maps:

$$\begin{aligned}\pi_{\leq s_0} &: \mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) \to \mathcal{I}(\mathbb{K}, \omega_{\leq s_0}, \mathbf{G}),\\ \pi^{\leq s_0}_{\leq s_0 - 1} &: \mathcal{I}(\mathbb{K}, \omega_{\leq s_0}, \mathbf{G}) \to \mathcal{I}(\mathbb{K}, \omega_{\leq s_0 - 1}, \mathbf{G}),\\ \pi_{\leq s_0 - 1} = \pi^{\leq s_0}_{\leq s_0 - 1} \circ \pi_{\leq s_0} &: \mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) \to \mathcal{I}(\mathbb{K}, \omega_{\leq s_0 - 1}, \mathbf{G}).\end{aligned}$$

Hence, for each $X \in \mathcal{I}(\mathbb{K}, \omega_{\leq s_0 - 1}, \mathbf{G})$, the map $\pi_{\leq s_0} : \mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) \to \mathcal{I}(\mathbb{K}, \omega_{\leq s_0}, \mathbf{G})$ induces a surjection:

$$\pi_{= s_0, X} : \pi^{-1}_{\leq s_0 - 1}(X) \to \left(\pi^{\leq s_0}_{\leq s_0 - 1}\right)^{-1}(X).$$

By using Step G again, we know that the set $\left(\pi^{\leq s_0}_{\leq s_0 - 1}\right)^{-1}(X)$ satisfies axioms (A1)-(A4) and (GG) with $n = 1$. By using Substep (i), we prove that it also satisfies axiom (CO) and hence it is an $\mathbb{R}$-building.

(iii) We prove that, whenever C and C' are two opposite Weyl chambers with common vertex x as in axiom (CO), if we set $X := \pi_{\leq s_0 - 1}(x)$, then the images of C and C' by $\pi_{= s_0, X}$ are opposite Weyl chambers with common vertex $\pi_{= s_0, X}(x)$ in the $\mathbb{R}$-building $\left(\pi^{\leq s_0}_{\leq s_0 - 1}\right)^{-1}(X)$.

(iv) By contradiction, we consider two opposite Weyl chambers C and C' with common vertex x as in axiom (CO) and we assume that they are not contained in a common apartment. By setting $X := \pi_{\leq s_0-1}(x)$ as before, we prove that there exists an integer $s_0 \in \{1,\dots,n\}$ such that their images by $\pi_{=s_0,X}$ are not contained in a common apartment in the $\mathbb{R}$-building $\left(\pi^{\leq s_0}_{\leq s_0-1}\right)^{-1}(X)$. Together with Substep (iii), this contradicts the fact that this $\mathbb{R}$-building satisfies axiom (CO).

7. Appendix: some axiomatic definitions

Axioms defining a Root Group Datum. Let G be a group and Φ be a root system. A root group datum of G of type Φ is a system $(T,(U_\alpha,M_\alpha)_{\alpha\in\Phi})$ satisfying the following axioms:

(RGD1) T is a subgroup of G and, for any root $\alpha\in\Phi$, the set U_α is a nontrivial subgroup of G, called the root group of G associated to α;

(RGD2) for any roots $\alpha,\beta\in\Phi$ such that $\beta\notin\mathbb{R}_{<0}\alpha$, the commutator subgroup $[U_\alpha,U_\beta]$ is contained in the subgroup generated by the root groups U_γ for $\gamma\in(\alpha,\beta)$;

(RGD3) if α is a multipliable root, we have $U_{2\alpha}\subset U_\alpha$ and $U_{2\alpha}\neq U_\alpha$;

(RGD4) for any root $\alpha\in\Phi$, the set M_α is a right coset of T in G and we have $U_{-\alpha}\setminus\{1\}\subset U_\alpha M_\alpha U_\alpha$;

(RGD5) for any roots $\alpha,\beta\in\Phi$ and any $m\in M_\alpha$, we have $mU_\beta m^{-1}=U_{r_\alpha(\beta)}$;

(RGD6) for any choice of positive roots Φ^+ on Φ, we have $TU^+\cap U^-=\{1\}$ where U^+ (resp. U^-) denotes the subgroup generated by the U_α for $\alpha\in\Phi^+$ (resp. $\alpha\in\Phi^-=-\Phi^+$).

We denote N the subgroup of G generated by the M_α for $\alpha\in\Phi$ if $\Phi\neq\emptyset$ and by $N=T$ otherwise. One can prove that axiom (RGD5) defines an epimorphism ${}^v\nu : N\to W(\Phi)$ such that ${}^v\nu(m)=r_\alpha$ for any $m\in M_\alpha$, any $\alpha\in\Phi$.

Axioms defining a Valued Root Group Datum. Let Φ be a root system and $(T,(U_\alpha,M_\alpha)_{\alpha\in\Phi})$ be a root group datum. An R_n-valuation of the root group datum is a family $(\varphi_\alpha)_{\alpha\in\Phi}$ of maps $\varphi_\alpha : U_\alpha\to R_n\cup\{\infty\}$ satisfying the following axioms:

(V0) for any $\alpha\in\Phi$, the set $\varphi_\alpha(U_\alpha)$ contains at least 3 elements;

(V1) for any $\alpha\in\Phi$ and $\lambda\in R_n\cup\{\infty\}$, the set $U_{\alpha,\lambda}=\varphi_\alpha^{-1}([\lambda,\infty])$ is a subgroup of U_α and $U_{\alpha,\infty}=\{1\}$;

(V2) for any $\alpha\in\Phi$ and $m\in M_\alpha$, the map $U_{-\alpha}\setminus\{1\}\to R_n$ defined by $u\mapsto\varphi_{-\alpha}(u)-\varphi_\alpha(mum^{-1})$ is constant;

(V3) for any $\alpha,\beta\in\Phi$ such that $\beta\notin\mathbb{R}_{\leqslant 0}\alpha$ and any $\lambda,\mu\in R_n$, the commutator group $[U_{\alpha,\lambda},U_{\beta,\mu}]$ is contained in the group generated by the $U_{r\alpha+s\beta,r\lambda+s\mu}$ for $r,s\in\mathbb{Z}_{>0}$ such that $r\alpha+s\beta\in\Phi$;

(V4) for any multipliable root $\alpha\in\Phi$, the map $\varphi_{2\alpha}$ is the restriction of the map $2\varphi_\alpha$ to $U_{2\alpha}$;

(V5) for any $\alpha \in \Phi$ and $u \in U_\alpha$, for any $u', u'' \in U_{-\alpha}$ such that $u'uu'' \in M_\alpha$, we have $\varphi_{-\alpha}(u') = -\varphi_\alpha(u)$.

Axioms defining an Action Compatible with a Valuation. Let Φ be a root system and $(T, (U_\alpha, M_\alpha)_{\alpha\in\Phi})$ be a root group datum together with an R_n-valuation $(\varphi_\alpha)_{\alpha\in\Phi}$. Let $\mathbb{A}$ be an R_n-affine space endowed with an origin o and let $\nu : N \to \mathrm{Aff}_{R_n}(\mathbb{A})$ be an action of N on $\mathbb{A}$ by R_n-affine endomorphisms. We say that the action of N on $\mathbb{A}$ is **compatible** with the valuation $(\varphi_\alpha)_{\alpha\in\Phi}$ if:

(CA1) the linear part of this action is equal to ${}^v\nu : N \to W(\Phi)$;

(CA2) for any $\alpha \in \Phi$ and any $u \in U_\alpha \setminus \{1\}$, we have $2\varphi_\alpha(u) + \alpha\Big(\nu\big(m(u)\big)(o) - o\Big) = 0$.

(CA3) for any $\alpha \in \Phi$ and $m \in M_\alpha$, the element $\nu(m)$ has order 2.

Here, $m(u)$ stands for the unique element in $M_{-\alpha}$ such that $u \in U_\alpha m(u) U_\alpha$. The existence and uniqueness of such an element is ensured by the root group datum axioms.

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