Bruhat–Tits theoretic approaches for Galois cohomology

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Contents

Chapter 1

Introduction and Convention

This is a note on Prasad's paper [Pra20] at TIMS.

1.1 Galois cohomology of number fields

Throughout this section, let K be a number field and let Ω_K be the set of places of K. Let G be an affine algebraic group over K. The Galois cohomology set $H^1(K, G)$ classifies G-torsors over K and controls certain local-global problems in arithmetic geometry. Thus it is an interesting question to study the pointed set $H^1(K, G)$.

One way to study $H^1(K, G)$ is to consider the following global-to-local map

$$\Delta : H^1(K, G) \to \prod_{v \in \Omega_K} H^1(K_v, G). \tag{1.1}$$

In the sequel, we briefly introduce several well-known results concerning the sets $H^1(K, G)$, $\operatorname{Ker} \Delta$ and $\operatorname{Coker} \Delta$. As a consequence, we will see that these sets indeed tell us something about the arithmetic of G over K.

1.1.1 The set $H^1(K, G)$

Conjecture 1.1.1 (Serre's conjecture II)**.** *Let G be a semi-simple simply connected group over a field F of* $\operatorname{cd}(F) \leq 2$ (*see* Definition 3.1.4 *below*). *Then $H^1(F, G) = 1$.*

In general, Serre's conjecture II is still open. The following theorem provides a first special case of Serre's conjecture II.

Theorem 1.1.2. *Suppose that K is totally imaginary. Let G be a semi-simple simply connected group over K. If G is quasi-split, Then $H^1(K, G) = 1$.*

Note that $\operatorname{cd}(K) = 2$ since K is totally imaginary. By quasi-split we mean that G contains a Borel subgroup defined over K. As of today, we can remove the assumption on quasi-splitness.

To understand how $H^1(K,G)$ is controlled, we begin with the local Galois cohomology sets. The following theorem provides a local version of Theorem 1.1.2 where we do drop the quasi-splitness assumption (which is quite strong).

Theorem 1.1.3. *Let G be a semi-simple simply connected group over K. For any finite place v, we have $H^1(K_v,G)=1$.*

In most literatures, the proof of this theorem is a Lie-theoretic case-by-case argument. During its original proof, people also obtained the following interesting result. We assume that G is absolutely almost-simple simply connected, so we can use the classification theory.

Theorem 1.1.4. *Let G be an absolutely almost-simple simply connected group over K_v for some finite place v. If G is K_v-anisotropic, then $G=\mathbf{SL}_n(D)$ for some finite dimensional central division K_v-algebra D. In other words, G is of type ${}^1\mathbf{A}$ (i.e., inner type $\mathbf{A}$).*

In this mini-course, we will give a proof of Theorem 1.1.4 and give a uniform proof of Theorem 1.1.3 using Bruhat–Tits building theory.

Let us go back to the investigation of the pointed set $H^1(K,G)$. In view of Theorem 1.1.3, we see that the global-to-local map (1.1) reduces to

$$\Delta : H^1(K,G) \to \prod_{v \text{ real}} H^1(K_v,G). \tag{1.2}$$

Theorem 1.1.5. *If G is a semi-simple simply connected group, then the global-to-local map Δ in (1.2) is bijective.*

Thus for an arbitrary number field K, the computation of $H^1(K,G)$ is completely controlled by the real cohomology $H^1(\mathbb{R},G)$. See [Bor88] for the computation of Galois cohomology of real reductive groups. Finally, if K is totally imaginary, then $H^1(K,G)=1$ is trivial by Theorem 1.1.5 without assuming G being quasi-split.

1.1.2 The kernel of the diagonal map

Since $H^1(K,G)$ classifies G-torsors over K, $\operatorname{Ker}\Delta$ classifies everywhere locally trivial G-torsors. We have seen $\operatorname{Ker}\Delta=1$ when G is a semi-simple simply group by Theorem 1.1.5. In general, we only ask whether $\operatorname{Ker}\Delta$ is finite.

Theorem 1.1.6 (Borel–Serre)**.** *Let G be an affine algebraic group over K. Then* $\operatorname{Ker}\Delta$ *is finite.*

If G is a connected reductive group over K, then $\operatorname{Ker}\Delta$ has a canonical abelian group structure. For example, see [San81, Théorème 8.5] and [Bor98, Theorem 5.13].

If we generalize the global-to-local map (1.1) to semi-global function fields (for example, $\mathbb{Q}_p(t)$), then the finiteness of $\operatorname{Ker}\Delta$ is still open.

1.1.3 The cokernel of the diagonal map

It is not obvious that $\operatorname{Coker}\Delta$ is interesting from its definition. However, Kottwitz proved that $\operatorname{Coker}\Delta$ can be described by the algebraic fundamental group $\pi_1^{\mathrm{alg}}(G_{\overline{K}})$ of G. This sequence together with some Poitou–Tate style arithmetic dualities is useful in comparing cohomological obstructions. See [Har02] for more details.

Theorem 1.1.7 (Kottwitz, [Bor98, Theorem 5.16])**.** *Let G be a connected reductive group over K. There is an exact sequence of pointed sets*

$$H^1(K,G) \to \prod_{v\in\Omega_K} H^1(K_v,G) \to (\pi_1^{\mathrm{alg}}((G_{\overline{K}})_{\mathrm{Gal}(\overline{K}|K)})_{\mathrm{tors}}.$$

1.2 Arithmetic setting of fields

From now on, we fix the following conventions.

(1) An arbitrary field will be systematically denoted by K.

(2) Let k be a field endowed with a **discrete** valuation v.

(3) Let $\overline{k}$ be a fixed algebraic closure of k.

(4) Let $\mathcal{O}$ be the valuation ring of v and let κ be the residue field. We assume throughout that $\mathcal{O}$ is **Henselian**.

(5) Let k_{nr} be the maximal unramified extension of k contained in $\overline{k}$. Let $\mathcal{O}_{\mathrm{nr}}$ be the valuation ring of k_{nr} and let κ_{sep} be the residue field of $\mathcal{O}_{\mathrm{nr}}$.

(6) Recall that κ_{sep} is the separable closure of κ. Let $\Gamma = \mathrm{Gal}(\kappa_{\mathrm{sep}}|\kappa) = \mathrm{Gal}(k_{\mathrm{nr}}|k)$ be the absolute Galois group of κ.

Example 1.2.1. Let $k = \mathbb{Q}_p$ be a p-adic field and let $v = v_p$ be any p-adic valuation. Then $\mathcal{O} = \mathbb{Z}_p$ is the ring of p-adic integers with residue field $\kappa = \mathbb{F}_p$. Moreover, k_{nr} is obtained by adjoining all roots of unity of order prime to p and $\mathrm{Gal}(k_{\mathrm{nr}}|k) = \mathrm{Gal}(\overline{\mathbb{F}}_p|\mathbb{F}_p) = \widehat{\mathbb{Z}}$ is the absolute Galois group of $\mathbb{F}_p$. So our settings indeed generalize the classical local field settings.

1.3 Statement of main theorems

Let k be a discrete valuation field with perfect residue field κ of $\mathrm{cd}(\kappa) \leq 1$. (For example, $\mathbb{Q}_p$ is such a field.) We shall give a proof of the following theorem generalizing Theorem 1.1.4 and Theorem 1.1.3 via the Bruhat–Tits building theory.

Theorem. *Let G be a semi-simple simply connected group over k.*

(1) *If G is absolutely almost-simple and k-anisotropic, then G is of type* $\mathbf{A}$.

(2) *Any G-torsor over k is trivial, i.e.,* $H^1(k, G) = 1$.

The first assertion is weaker than Theorem 1.1.4. However, if k is a local non-Archimedean field, then G is of inner type $\mathbf{A}$ provided that G is absolutely almost-simple and k-anisotropic. This was proved by Kneser in char 0 and by Bruhat–Tits in general. For this purpose, it will be sufficient to show that such groups G of outer type $\mathbf{A}_n$ are isotropic for $n \geq 2$. Roughly speaking, the group $G = \mathbf{SU}(h)$ is isotropic if and only if h represents zero. However, any Hermitian form in $n + 1 \geq 3$ variables over a non-Archimedean local field automatically represents zero. Thus G is isotropic.

1.4 Geometric and group-theoretic settings

We keep the following notations and conventions from now on.

(1) A K-variety is a septated K-scheme of finite type.

(2) For any k-variety X, we shall simply write $X_{\mathrm{nr}} := X \times_k k_{\mathrm{nr}}$.

(3) Varieties defined over k_{nr} will be systematically denoted by $\widetilde{X}$.

(4) For any k-torus T and any algebraic field extension $K|k$, let

$$\mathbf{X}^*(K, T) := \mathrm{Hom}_K(T_K, \mathbb{G}_m) \quad \text{and} \quad \mathbf{X}_*(K, T) := \mathrm{Hom}_K(\mathbb{G}_m, T_K).$$

We keep the following conventions about algebraic groups.

(1) Let $\mathfrak{G}$ an arbitrary connected smooth affine algebraic k-group.

(2) The centre of $\mathfrak{G}$ is denoted by $Z(\mathfrak{G})$.

(3) Let $\mathscr{D}\mathfrak{G}$ be the derived subgroup of $\mathfrak{G}$.

(4) Let G be a connected reductive group over k.

(5) Let $G^{\mathrm{ss}} := \mathscr{D}G$ be the derived subgroup of G, which is a semi-simple group.

(6) Let $G^{\mathrm{sc}} \to G^{\mathrm{ss}}$ be the universal covering of G^{ss}, which is a simply connected group.

(7) Let $C_G(T)$ (resp. $N_G(T)$) be the centralizer (resp. normalizer) of a torus T in G.

Recall that $Z(G)$ is an affine algebraic group of multiplicative type and that $Z(G^{\mathrm{ss}})$ is a finite algebraic group (of multiplicative type).

Apartments, chambers and facets in $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ will be written as A, C and F, respectively. Those in $\mathcal{B} := \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})^{\Gamma}$ will be written as $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{F}$, respectively.

Finally, we recall definitions on quasi-split algebraic groups.

Definition 1.4.1. A **Borel subgroup** of $\mathfrak{G}$ is a connected smooth solvable (affine) subgroup B such that $\mathfrak{G}/B$ is complete. The group $\mathfrak{G}$ is **quasi-split** if it contains a Borel subgroup defined over K.

Example 1.4.2. The group $\mathbf{GL}_n$ is quasi-split because all upper triangular matrices form a Borel subgroup.

1.5 Pseudo-reductive groups

Let $\mathfrak{G}$ be a connected smooth affine algebraic group over a field K.

Definition 1.5.1. The **K-unipotent radical** $\mathrm{rad}^u(\mathfrak{G})$ of $\mathfrak{G}$ is the largest connected smooth unipotent normal K-subgroup of $\mathfrak{G}$. The group $\mathfrak{G}$ is **pseudo-reductive** if $\mathrm{rad}^u(\mathfrak{G}) = 1$.

Recall that $\mathfrak{G}$ is reductive if $\mathrm{rad}^u(\mathfrak{G}_{\overline{K}}) = 1$. But the formation of $\mathrm{rad}^u(\mathfrak{G})$ commutes with separable field extension, so pseudo-reductive groups are reductive if K is perfect.

Definition 1.5.2.

(1) Let X be a K-variety and let $\varphi : \mathbb{G}_m \to X$ be a morphism of K-varieties. If φ can be extended to a morphism $\widetilde{\varphi} : \mathbb{A}^1 \to X$, then we say that $\lim_{t\to 0} \varphi(t)$ exists.

(2) Let $\mathfrak{G}$ be an affine algebraic group and take any $\lambda \in \mathbf{X}_*(K, \mathfrak{G})$. We define a functor

$$P_{\mathfrak{G}}(\lambda) : \mathfrak{Alg}_K \to \mathfrak{Gp},\ R \mapsto \{g \in \mathfrak{G}(R) \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}.$$

(3) Let $\mathfrak{G}$ be a connected smooth affine algebraic group over K. We say that $\mathfrak{G}$ is **pseudo-parabolic** if $\mathfrak{G} = P_{\mathfrak{G}}(\lambda)\,\mathrm{rad}^u(\mathfrak{G})$.

Remark 1.5.3.

(1) A smooth affine algebraic subgroup P of $\mathfrak{G}$ is **parabolic** if $\mathfrak{G}/P$ is a complete variety. Similarly to pseudo-reductive groups, pseudo-parabolic subgroups are parabolic if the base field K is perfect.

(2) The group $\mathfrak{G}$ contains a proper parabolic subgroup if and only if it is isotropic ([Mil17, Proposition 25.2]). Moreover, every parabolic subgroup of a connected reductive group $\mathfrak{G}$ is of the form $P_{\mathfrak{G}}(\lambda)$ for some $\lambda \in \mathbf{X}_*(K, \mathfrak{G})$ by [Mil17, Theorem 25.1].

(3) If $\mathfrak{G}$ is quasi-split, then Borel subgroups are (minimal) parabolic subgroups.

(4) We give an example of a parabolic subgroup of $\mathbf{GL}_4$ as follows

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}.$$

Chapter 2

Bruhat–Tits Theory

Let k be a discrete valuation field as in Section 1.2. In this chapter, we **assume** systematically that the Bruhat–Tits theory is available for any connected reductive group over k_{nr}. More precisely, there is an **affine building** $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$, called **the Bruhat–Tits building of** $G(k_{\mathrm{nr}})$ (also of $G^{\mathrm{ss}}(k_{\mathrm{nr}})$, see Hypothesis 2.1.3 below), such that

(1) the group $G(k_{\mathrm{nr}})$ acts on $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ by polysimplicial automorphisms, and

(2) given a non-empty bounded subset Ω of an apartment of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$, there is a smooth affine $\mathcal{O}_{\mathrm{nr}}$-group scheme $\mathscr{G}_\Omega^\circ$ with generic fibre G_{nr} and connected special fibre.

Thanks to [BT84, §4], these assumptions are fulfilled if G is quasi-split over k_{nr}. For instance, when the residue field κ is perfect and $\mathrm{cd}(\kappa) \leq 1$, then G is quasi-split over k_{nr} by Steinberg's theorem (Theorem 3.1.7).

In this chapter, the purpose is to establish a Bruhat–Tits theory over k from that over k_{nr} (i.e., Bruhat–Tits theory descends from the maximal unramified extension to the base). In the first section, we make some **assumptions**[1] and deduce some straightforward consequences. Subsequently, we derive a Bruhat–Tits theory over k.

2.1 Preliminaries

Recall that we have assumed that Bruhat–Tits theory is available for G over k_{nr}.

2.1.1 The enlarged Bruhat–Tits building

Let $\mathfrak{Z} \subset Z(G)$ be the maximal k-torus splitting over k_{nr}. Consider the natural Γ-action on $\mathbf{X}_*(k_{\mathrm{nr}}, \mathfrak{Z})$. Note that

$$\mathbf{X}_*(k_{\mathrm{nr}}, \mathfrak{Z})^\Gamma = \mathrm{Hom}_k(\mathbb{G}_m, \mathfrak{Z}) = \mathbf{X}_*(k, \mathfrak{Z}).$$

[1] These assumptions are indeed valid over local fields. See [BT72, BT84] for more information.

We put

$$V(\mathfrak{Z}_{\mathrm{nr}}) := \mathbf{X}_*(k_{\mathrm{nr}}, \mathfrak{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The Γ-action on $\mathbf{X}_*(k_{\mathrm{nr}}, \mathfrak{Z})$ extends to an $\mathbb{R}$-linear action on $V(\mathfrak{Z}_{\mathrm{nr}})$, and

$$V(\mathfrak{Z}_{\mathrm{nr}})^{\Gamma} = \mathbf{X}_*(k, \mathfrak{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Definition 2.1.1. Let $\mathcal{B}(G/k_{\mathrm{nr}}) := V(\mathfrak{Z}_{\mathrm{nr}}) \times \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. We define it as the **enlarged Bruhat–Tits building** of $G(k_{\mathrm{nr}})$.

Remark 2.1.2. If G is semi-simple, then $Z(G)$ is a finite group of multiplicative type over k. In particular, the torus $\mathfrak{Z}$ is trivial and $V(\mathfrak{Z}_{\mathrm{nr}}) = 0$. On the other hand, $G = G^{\mathrm{ss}}$ since it is semi-simple. Hence there is no ambiguity on the notation $\mathcal{B}(G/k_{\mathrm{nr}}) = \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.

Hypothesis 2.1.3. If $\mathfrak{G} \to \mathfrak{H}$ is a homomorphism of connected reductive K-groups with central kernel, then there is an isomorphism $\mathcal{B}(\mathfrak{G}/K) \to \mathcal{B}(\mathfrak{H}/K)$ of affine buildings.

Therefore the affine building $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ of G can be identified with that of

- the adjoint group G^{ad} via $1 \to Z(G) \to G \to G^{\mathrm{ad}} \to 1$,
- its derived subgroup G^{ss} via $G^{\mathrm{ad}} = G/Z(G) = G^{\mathrm{ss}}/Z(G^{\mathrm{ss}}) = (G^{\mathrm{ss}})^{\mathrm{ad}}$, and
- the simply connected covering G^{sc} of G^{ss} via the central isogeny $G^{\mathrm{sc}} \to G^{\mathrm{ss}}$.

2.1.2 Description of building axioms

Axioms on group actions

By assumption, there is a $G(k_{\mathrm{nr}})$-action on the affine building $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. The behavior of the $G(k_{\mathrm{nr}})$-action can be described via the following hypothesis.

Hypothesis 2.1.4.

(1) We assume that there is a $G(k_{\mathrm{nr}})$-equivariant **bijective correspondence**

$$\{\text{apartments of } \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})\} \leftrightarrow \{\text{maximal } k_{\mathrm{nr}}\text{-split torus of } G_{\mathrm{nr}}\}. \tag{2.1}$$

(2) Let $\widetilde{T}$ be a maximal k_{nr}-split torus of G_{nr} with corresponding apartment A. Under the above correspondence, A is an affine space under $V(\widetilde{T}) := \mathbf{X}_*(k_{\mathrm{nr}}, \widetilde{T}) \otimes_{\mathbb{Z}} \mathbb{R}$.

(3) Consider the set Θ of ordered pairs (A, C), where A is an apartment of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ and C is a chamber in A. The group $G^{\mathrm{ss}}(k_{\mathrm{nr}})$ acts transitively on Θ.

Construction 2.1.5. We define a $G(k_{\mathrm{nr}})$-action on the set of apartments of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. Let $\widetilde{T}$ be a maximal k_{nr}-split torus of G_{nr} (over k_{nr}) and let A be the apartment of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ corresponding to $\widetilde{T}$.

- For any $g \in G(k_{\mathrm{nr}})$, we define $g \cdot A$ to be the apartment corresponding to $g\widetilde{T}g^{-1}$. Thus

$$g \cdot A = A \iff g \in N_{G_{\mathrm{nr}}}(\widetilde{T})(k_{\mathrm{nr}}).$$

- In particular, $N_{G_{\mathrm{nr}}}(\widetilde{T})(k_{\mathrm{nr}}) \cap G^{\mathrm{ss}}(k_{\mathrm{nr}})$ acts transitively on the set of chambers in A.

Axioms on Bruhat–Tits group schemes

We describe some basic properties of the $\mathcal{O}_{\mathrm{nr}}$-group scheme $\mathscr{G}_\Omega^\circ$. For simplicity, in this subsubsection we assume throughout that G is **semi-simple**. We keep the notation $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ to emphasize that these properties are still valid if G is a connected reductive group.[2]

Hypothesis 2.1.6. Let Ω be a non-empty bounded subset of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.

(1) There is a smooth affine $\mathcal{O}_{\mathrm{nr}}$-group scheme $\mathscr{G}_\Omega$ such that

$$\mathscr{G}_\Omega(\mathcal{O}_{\mathrm{nr}}) = \{g \in G(k_{\mathrm{nr}}) \mid g \cdot x = x \text{ for all } x \in \Omega\}. \tag{2.2}$$

(2) Let $\mathscr{G}_\Omega^\circ$ be the neutral component of $\mathscr{G}_\Omega$. Thus it is the union of the connected generic fibre G_{nr} and the neutral component of the special fibre of $\mathscr{G}_\Omega$. We assume that $\mathscr{G}_\Omega^\circ$ is an affine open $\mathcal{O}_{\mathrm{nr}}$-subgroup scheme of $\mathscr{G}_\Omega$.

We may change the bounded subset Ω of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$, we make the following hypothesis.

Hypothesis 2.1.7. Let Ω be a non-empty bounded subset of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.

(1) Let $\overline{\Omega}$ be the closure of Ω in $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. Then $\mathscr{G}_{\overline{\Omega}}^\circ = \mathscr{G}_\Omega^\circ$.

(2) Let F be a facet of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ containing Ω. Then $\mathscr{G}_\Omega^\circ = \mathscr{G}_F^\circ$.

(3) If G is semi-simple simply connected and quasi-split over k_{nr}, then $\mathscr{G}_\Omega^\circ = \mathscr{G}_\Omega$.

(4) Under the same condition as (3), we assume

$$\mathrm{Stab}_{G(k_{\mathrm{nr}})}(\Omega) = \mathscr{G}_\Omega^\circ(\mathcal{O}_{\mathrm{nr}}).$$

Concerning closed subtorus of the smooth affine group scheme $\mathscr{G}_\Omega^\circ$, we make the following assumption.

Hypothesis 2.1.8. Let A be an apartment of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ with corresponding maximal k_{nr}-split torus $\widetilde{T}$ of G_{nr}. We assume that there is a **closed $\mathcal{O}_{\mathrm{nr}}$-split torus $\mathscr{T}$** in $\mathscr{G}_\Omega^\circ$ with generic fibre $\widetilde{T}$ such that the special fibre $\mathscr{T}_{\kappa_{\mathrm{sep}}}$ of $\mathscr{T}$ is a maximal κ_{sep}-split torus of $\mathscr{G}^\circ_{\Omega,\kappa_{\mathrm{sep}}}$. This can be visualized by the diagram

$$\begin{tikzcd}
G_{\mathrm{nr}} & \widetilde{T} \arrow[l, hook', "\text{maximal}"', "k_{\mathrm{nr}}\text{-split}"] \arrow[r] \arrow[d] & \mathscr{T} \arrow[d] & \mathscr{T}_{\kappa_{\mathrm{sep}}} \arrow[l] \arrow[d] \arrow[r, hook, "\text{maximal}", "k_{\mathrm{nr}}\text{-split}"'] & \mathscr{G}^\circ_{\Omega,\kappa_{\mathrm{sep}}} \\
& \mathrm{Spec}\, k_{\mathrm{nr}} \arrow[r] & \mathrm{Spec}\, \mathcal{O}_{\mathrm{nr}} & \mathrm{Spec}\, \kappa_{\mathrm{sep}}. \arrow[l] &
\end{tikzcd}$$

Definition 2.1.9 (A partial order)**.** Let $\Omega, \Omega_0 \subset \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ be non-empty bounded subsets. We put

$$\Omega_0 \prec \Omega \text{ if and only if } \Omega_0 \subset \overline{\Omega}.$$

In particular, we may talk about maximal facets (chambers) and minimal facets (vertices).

[2] See [Pra20, pp. 222, last paragraph] for the construction of $\mathscr{G}_\Omega$ for general connected reductive groups.

If we are given two facets $F_0 \prec F$, we would like to compare $\mathscr{G}_{F_0}^\circ$ and $\mathscr{G}_F^\circ$. To this end, we begin with two non-empty bounded subsets $\Omega_0 \prec \Omega$. The RHS of (2.2) tells us that there is an inclusion $\mathscr{G}_\Omega(\mathcal{O}_{\mathrm{nr}}) \subset \mathscr{G}_{\Omega_0}(\mathcal{O}_{\mathrm{nr}})$. We **assume** that there is an $\mathcal{O}_{\mathrm{nr}}$-group scheme homomorphism

$$\rho_{\Omega_0,\Omega} : \mathscr{G}_\Omega \to \mathscr{G}_{\Omega_0}$$

which is the identity on the generic fibre G_{nr}. We thus obtain an $\mathcal{O}_{\mathrm{nr}}$-group scheme homomorphism $\rho_{\Omega_0,\Omega} : \mathscr{G}_\Omega^\circ \to \mathscr{G}_{\Omega_0}^\circ$ and a κ_{sep}-group scheme homomorphism $\overline{\rho}_{\Omega_0,\Omega} : \mathscr{G}_{\Omega,\kappa_{\mathrm{sep}}}^\circ \to \mathscr{G}_{\Omega_0,\kappa_{\mathrm{sep}}}^\circ$ on special fibres.

Hypothesis 2.1.10. Let $F_0 \prec F$ be two facets of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.

(1) We denoted by $\mathfrak{p}(F_0|F) := \mathrm{Im}\,\overline{\rho}_{F_0,F}$ which is a subgroup of $\mathscr{G}_{F_0,\kappa_{\mathrm{sep}}}^\circ$. We assume that $\mathfrak{p}(F_0|F)$ is **pseudo-parabolic**.

(2) Thus we obtain an order-preserving map

$$\{F \,|\, F_0 \prec F\} \to \{\text{pseudo-parabolic } \kappa_{\mathrm{sep}}\text{-subgroup of } \mathscr{G}_{F_0,\kappa_{\mathrm{sep}}}^\circ\},\ F \mapsto \mathfrak{p}(F_0|F),$$

where the latter set is partially-ordered by **opposite** of inclusion. We **assume** that this map is bijective.

(3) Consider the canonical projection

$$\pi_{F_0} : \mathscr{G}_{F_0}^\circ(\mathcal{O}_{\mathrm{nr}}) \to \mathscr{G}_{F_0,\kappa_{\mathrm{sep}}}^\circ(\kappa_{\mathrm{sep}}).$$

We **assume** that the inverse image of the subgroup $\mathfrak{p}(F_0|F)(\kappa_{\mathrm{sep}})$ under π_{F_0} is $\mathscr{G}_F^\circ(\mathcal{O}_{\mathrm{nr}})$.

Applying Hypothesis 2.1.10(3) to a facet F yields the following corollary.

Corollary 2.1.11. *Let F be a facet of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. Then chambers containing F are in one-to-one correspondence with minimal pseudo-parabolic κ_{sep}-subgroups of $\mathscr{G}_F^\circ$.*

2.1.3 Bounded subsets

Bounded subgroups are useful when we study special k-apartments. Before going further, we take a quick review on bounded subgroups.

Definition 2.1.12. A subset $\Theta \subset G(k)$ is bounded, if the following set is bounded below

$$\{v(f(x)) \,|\, \text{for any } f \in k[G],\, x \in \Theta\}.$$

Since G is an affine algebraic group, it can be realized as a closed subvariety of some affine space $\mathbb{A}^n$. Then $\Theta \subset G(k)$ is bounded if and only if its image in $\mathbb{A}^n(k) = k^n$ is bounded.

The k-rank of G is the dimension of a maximal k-split torus in G. The connected reductive group G is anisotropic if the k-rank of G equals zero. Otherwise, we say that G is isotropic.

Theorem 2.1.13. *The G is k-anisotropic if and only if group $G(k)$ is bounded.*

Proof. See [Pra20, Theorem 1.1]. □

If k is a non-discrete locally compact field (for instance, $\mathbb{Q}_p$), then $G(k)$ is compact with respect to the analytic topology induced by k if and only if G is k-anisotropic.

Proposition 2.1.14. *Suppose that G^{ss} is k-anisotropic. Then $G(k)$ contains a unique maximal bounded subgroup $G(k)_{\mathrm{bdd}}$. More precisely, it is given by*

$$G(k)_{\mathrm{bdd}} := \{g \in G(k) \,|\, \chi(g) \in \mathcal{O}^\times \textit{ for all } \chi \in \mathbf{X}^*(k, G)\}.$$

Proof. See [Pra20, Proposition 1.3]. □

Example 2.1.15. Let S be a maximal k-split torus of G and let $C_G(S)$ be the centralizer of S in G. Then $\mathscr{D}(C_G(S))$ is k-anisotropic since S is a maximal k-split torus of G. In particular, $C_G(S)(k)$ contains a unique maximal bounded subgroup. More precisely, we have

$$C_G(S)(k)_{\mathrm{bdd}} = \{z \in C_G(S)(k) \,|\, \chi(z) \in \mathcal{O}^\times \text{ for all } \chi \in \mathbf{X}^*(k, C_G(S))\}.$$

2.1.4 Special k-objects

In this subsection, we pass from affine building structure over k_{nr} to that over k. This procedure means exactly "unramified descent".

Galois actions

We have described the $G(k_{\mathrm{nr}})$-action on $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. Then Γ acts naturally on the affine building $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ by polysimplicial isometries. Moreover, the Γ-action on $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ is compatibe with the $G(k_{\mathrm{nr}})$-action. More precisely, the Γ-action is supposed to satisfy the following conditions.

(1) The orbit $\{{}^\sigma x \,|\, \sigma \in \Gamma\}$ is finite for any $x \in \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.

(2) (Compatibility, [BT84, 4.2.12]). For any $g \in G(k_{\mathrm{nr}})$, $x \in \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$, $\sigma \in \Gamma$, we have

$${}^\sigma(g \cdot x) = {}^\sigma g \cdot {}^\sigma x.$$

Unramified descents

Let Ω be a non-empty bounded subset of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. If Ω is Γ-stable, then the $\mathcal{O}_{\mathrm{nr}}$-group schemes $\mathscr{G}_\Omega$ and $\mathscr{G}_\Omega^\circ$ descend uniquely to smooth affine $\mathcal{O}$-group schemes. By abuse of notation, we still write $\mathscr{G}_\Omega$ and $\mathscr{G}_\Omega^\circ$ for the resulting $\mathcal{O}$-group schemes.

Similarly, the $\mathcal{O}_{\mathrm{nr}}$-torus $\mathscr{T}$ also descends uniquely to a closed $\mathcal{O}$-torus of $\mathscr{G}_\Omega^\circ$, which will be denoted by $\mathscr{T}$ as well. Then generic fibre $\mathscr{T}_k$ of $\mathscr{T}$ is T and the special fibre $\mathscr{T}_\kappa$ of $\mathscr{T}$ is a maximal κ-torus of $\mathscr{G}_{\Omega,\kappa}^\circ$.

Definition of special k-objects

Since Γ acts on $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$, we are allowed to define $\mathcal{B} := \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})^{\Gamma}$. Note that $\mathcal{B}$ is closed and convex in $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. Moreover, $\mathcal{B}$ is stable under the $G(k)$-action on $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.

Definition 2.1.16.

(1) A facet F of the building $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ is a **k-facet**, if $F \cap \mathcal{B} \neq \varnothing$.

(2) Maximal k-facets (with respect to $\prec$) of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ are called **k-chambers**.

(3) A k-torus $T \subset G$ is a **special k-torus**, if it contains a maximal k-split torus of G and $T_{\mathrm{nr}} \subset G_{\mathrm{nr}}$ is a maximal k_{nr}-split torus.

(4) A **special k-apartment** of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ is the apartment corresponding to T_{nr} by (2.1) for some special k-torus T.

Remark 2.1.17. A facet F of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ is a k-facet if and only if F is Γ-stable.

Existence of special k-tori

We show that $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ does contain special k-apartments. By definition, it suffices to prove that G contains special k-tori.

Lemma 2.1.18. *Let $\mathscr{G}$ be a smooth affine $\mathcal{O}$-group scheme.*

(1) *Let $\overline{\mathscr{T}} \subset \mathscr{G}_\kappa$ be a κ-torus. There exists a closed $\mathcal{O}$-torus $\mathscr{T} \subset \mathscr{G}$ such that $\mathscr{T}_\kappa \simeq \overline{\mathscr{T}}$.*

(2) *Let $\mathscr{T}, \mathscr{T}' \subset \mathscr{G}$ be two closed $\mathcal{O}$-tori such that there is an element $\overline{g} \in \mathscr{G}_\kappa(\kappa)$ such that $\overline{g}\mathscr{T}_\kappa\overline{g}^{-1} = \mathscr{T}'_\kappa$. There exists $g \in \mathscr{G}(\mathcal{O})$ lying over $\overline{g}$ such that $g\mathscr{T}g^{-1} = \mathscr{T}'$.*

(3) *Let $\mathscr{T}$ be a closed $\mathcal{O}$-torus of $\mathscr{G}$. The normalizer $N_{\mathscr{G}}(\mathscr{T})$ is a closed smooth $\mathcal{O}$-subgroup scheme of $\mathscr{G}$. In particular, the natural homomorphism below is surjective*

$$N_{\mathscr{G}}(\mathscr{T})(\mathcal{O}) \to N_{\mathscr{G}}(\mathscr{T})(\kappa).$$

Proof. See [Pra20, Proposition 2.1]. □

Proposition 2.1.19. *Any connected reductive group G contains a special k-torus.*

We use Lemma 2.1.18(1) to produce an $\mathcal{O}$-torus $\mathscr{T}$ whose generic fibre is a special k-torus.

Proof. Let S be a maximal k-split torus of G and $\mathscr{S}$ be the split $\mathcal{O}$-torus with generic fibre S. Then $\mathscr{S}(\mathcal{O}_{\mathrm{nr}}) \subset G(k_{\mathrm{nr}})$ is a maximal bounded subgroup. According to the Bruhat–Tits fixed point theorem [BT72, Proposition 3.2.4], there exists $x \in \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ fixed under the $\mathscr{S}(\mathcal{O}_{\mathrm{nr}})$-action. By Galois theory of valuations, $\mathscr{S}(\mathcal{O}_{\mathrm{nr}})$ is Γ-stable. It follows that x is fixed by Γ as well, i.e., $x \in \mathcal{B} = \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})^{\Gamma}$.

Since $\mathscr{S}(\mathcal{O}_{\mathrm{nr}})$ fixes x, there is a natural inclusion $\mathscr{S}(\mathcal{O}_{\mathrm{nr}}) \to \mathscr{G}_x(\mathcal{O}_{\mathrm{nr}})$ (see (2.2)) which yields an $\mathcal{O}$-group scheme homomorphism $\varphi : \mathscr{S} \to \mathscr{G}_x$ being the inclusion $S \to G$ on the generic fibre (see [BT84, Proposition 1.7.6]). We identify $\mathscr{S}$ with a closed $\mathcal{O}$-torus of $\mathscr{G}_x$ via φ.

Let $\mathscr{M} := C_{\mathscr{G}_x}(\mathscr{S})$ which is a smooth $\mathcal{O}$-group scheme by [SGA3II, Exposé XI, Corollary 5.3]. Applying Lemma 2.1.18 to $\mathscr{M}$ yields a closed $\mathcal{O}$-torus $\mathscr{T} \subset \mathscr{M}$ such that $\mathscr{T}_\kappa$ is a maximal κ-torus of $\mathscr{M}_\kappa$. Therefore $\mathscr{T}$ contains $\mathscr{S}$ (it is a central torus in $\mathscr{M}$) and it follows that $S \simeq \mathscr{S}_k$ is contained in $T := \mathscr{T}_k$. By construction, $\mathscr{T}_\kappa \subset \mathscr{M}_\kappa$ is a maximal torus of $\mathscr{G}_{x,\kappa}$ which splits automatically over κ_{sep}. Subsequently, T_{nr} is a maximal k_{nr}-split torus of G_{nr} as $\mathcal{O}_{\mathrm{nr}}$ is a Henselian discrete valuation ring. (See also [CTHH+22, Proposition A.1].) ☐

Properties of k-chambers

We want to show that special k-apartments contain k-chambers. First of all, we would like to find a criterion on whether a bounded subset is contained in a given apartment.

Lemma 2.1.20. *Let T, T_0 be maximal k_{nr}-split tori of G_{nr} and let A, A_0 be the corresponding apartments. Let $\Omega \subset A$ be a non-empty bounded subset. Then $\Omega \subset A_0$ if and only if one of the following equivalent conditions hold:*

(1) *There is an element $g \in \mathscr{G}^\circ_\Omega(\mathcal{O}_{\mathrm{nr}})$ such that $T_0 = gTg^{-1}$.*

(2) *The group scheme $\mathscr{G}^\circ_\Omega$ contains a closed $\mathcal{O}_{\mathrm{nr}}$-torus with generic fibre T_0.*

(3) *The intersection $\mathscr{G}^\circ_\Omega(\mathcal{O}_{\mathrm{nr}}) \cap T_0(k_{\mathrm{nr}})$ is the maximal bounded subgroup of $T_0(k_{\mathrm{nr}})$.*

Proof. See [Pra20, Proposition 2.2]. ☐

To proceed, we need to set forth the relation between k-chambers and pseudo-parabolic κ-subgroups. Compare Hypothesis 2.1.10 over κ_{sep} with Remark 2.1.21 below over κ.

Remark 2.1.21. Let $\Omega_0 \prec \Omega \subset \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ be Γ-stable non-empty bounded subsets. Then $\rho_{\Omega_0,\Omega} : \mathscr{G}^\circ_\Omega \to \mathscr{G}^\circ_{\Omega_0}$ **descends** to an $\mathcal{O}$-group scheme homomorphism that is identity on the generic fibre G.

(1) Let $F_0 \prec F$ be two k-facets of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. Then the image $\mathfrak{p}(F_0|F)$ of $\mathscr{G}^\circ_{F,\kappa} \to \mathscr{G}^\circ_{F_0,\kappa}$ is a pseudo-parabolic κ-subgroup of $\mathscr{G}^\circ_{F_0,\kappa}$.

(2) There is an order-preserving bijective map

$$\{F \mid F_0 \prec F\} \to \{\text{pseudo-parabolic } \kappa\text{-subgroup of } \mathscr{G}^\circ_{F_0,\kappa}\},\ F \mapsto \mathfrak{p}(F_0|F).$$

Here the latter set is partially-ordered by opposite of inclusion. Note also that the canonical projection $\mathscr{G}^\circ_{F_0,\kappa} \to \mathscr{G}^\circ_{F_0,\kappa}/\operatorname{rad}^u(\mathscr{G}^\circ_{F_0,\kappa})$ induces an inclusion preserving bijection

$$\{\text{pseudo-parabolic subgroups of } \mathscr{G}^\circ_{F_0,\kappa}\} \to \{\text{those of } \mathscr{G}^\circ_{F_0,\kappa}/\operatorname{rad}^u(\mathscr{G}^\circ_{F_0,\kappa})\}.$$

(3) Thus C is a k-chamber containing a given k-facet F_0 if and only if $\mathfrak{p}(F_0|C)$ is a minimal pseudo-parabolic κ-subgroup of $\mathscr{G}^\circ_{F_0,\kappa}$.

(4) Since a sequence $F_0 \prec F_1 \prec \cdots \prec F_n$ of k-facets corresponds to a decreasing sequence $\mathfrak{p}(F_0|F_1) \supset \cdots \supset \mathfrak{p}(F_0|F_n)$ of pseudo-parabolic κ-subgroups of $\mathscr{G}^\circ_{F_0,\kappa}$, we see that $\operatorname{codim}(F_0 \cap \mathcal{B}, \mathcal{B})$ is the κ-rank of the derived subgroup of $\mathscr{G}^\circ_{F_0,\kappa}$.

Remark 2.1.21 tells us when a k-facet is a k-chamber. We also have the following criterion.

Lemma 2.1.22 ([CGP15, Lemma 2.2.3])**.** *A k-facet F is a k-chamber if and only if the pseudo-reductive group $\mathscr{G}^\circ_{F,\kappa}/\operatorname{rad}^u(\mathscr{G}^\circ_{F,\kappa})$ contains a unique maximal κ-split torus.*

Consequently, the group scheme $\mathscr{G}^\circ_F$ plays an important role in Bruhat–Tits theory. This leads to the following definition.

Definition 2.1.23. For a facet F of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$, $\mathscr{G}^\circ_F$ (resp. $\mathscr{G}^\circ_F(\mathcal{O}_{\mathrm{nr}})$) is called the **Bruhat–Tits parahoric group scheme** (resp. the **parahoric subgroup of $G(k_{\mathrm{nr}})$**).

Proposition 2.1.24. *Let A be any special k-apartment of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. Then A contains a k-chamber.*

Proof. Let T be the corresponding special k-torus. By definition, T contains a maximal k-split torus $S \subset G$ and T_{nr} is a maximal k_{nr}-split torus of G_{nr}. Since A is Γ-stable, it contains a point $x \in \mathcal{B}$ by the Bruhat–Tits fixed point theorem. Let F be a facet of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ contained in A and containing x. As $x \in \mathcal{B}$, F is a k-facet.

- We find a minimal pseudo-parabolic κ-subgroup of $\mathscr{G}^\circ_{F,\kappa}$. Let $\mathscr{T} \subset \mathscr{G}^\circ_F$ be the closed $\mathcal{O}$-torus with generic fibre T and let $\mathscr{S} \subset \mathscr{T}$ be the maximal $\mathcal{O}$-split subtorus. Thus $\mathscr{S}_k = S$ by construction. We fix a minimal pseudo-parabolic κ-subgroup $\overline{\mathscr{P}}$ of $\mathscr{G}^\circ_{F,\kappa}$ containing $\mathscr{S}_\kappa$. Note that $\mathscr{T}_\kappa \subset \overline{\mathscr{P}}$ by [CGP15, Proposition C.2.4].

- We proceed a k-facet C from $\overline{\mathscr{P}}$. Let $\mathcal{P}$ be the inverse image of $\overline{\mathscr{P}}(\kappa_{\mathrm{sep}})$ in $\mathscr{G}^\circ_F(\mathcal{O}_{\mathrm{nr}})$ under the natural homomorphism $\mathscr{G}^\circ_F(\mathcal{O}_{\mathrm{nr}}) \to \mathscr{G}^\circ_{F,\kappa}(\kappa_{\mathrm{sep}})$. Then $\mathcal{P} \subset \mathscr{G}^\circ_F(\mathcal{O}_{\mathrm{nr}})$ is a parahoric subgroup of $G(k_{\mathrm{nr}})$ by Hypothesis 2.1.10(4). Moreover, $\mathscr{T}(\mathcal{O}_{\mathrm{nr}}) \subset \mathcal{P}$ by $\mathscr{T}_\kappa \subset \overline{\mathscr{P}}$ and $\mathcal{P}$ is stable under the Γ-action on $G(k_{\mathrm{nr}})$. Let C be the facet of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ fixed by $\mathcal{P}$. Since $F \prec C$ and C is Γ-stable, C is a k-facet.

- We show that C is as desired. Since $\overline{\mathscr{P}}$ is a minimal pseudo-parabolic κ-subgroup of $\mathscr{G}^\circ_{F,\kappa}$, it is a k-chamber. Finally, since $\mathscr{T}(\mathcal{O}_{\mathrm{nr}}) \subset \mathcal{P}$, we see that C indeed lies in A by Lemma 2.1.20. ∎

2.2 Affine building of $G(k)$

In this section, we will show that $\mathcal{B} := \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})^\Gamma$ has an affine building structure.

2.2.1 Polysimplicial complex structure

We define facets, chambers and apartments in $\mathcal{B}$. In the sequel, we show that these constructions make $\mathcal{B}$ into an affine building.

Construction 2.2.1. We endow $\mathcal{B} := \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})^{\Gamma}$ with a polysimplicial complex structure.

- **Facets** of $\mathcal{B}$ are of the form $F \cap \mathcal{B}$, where F runs over Γ-stable facets of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.
- **Faces**. If $\mathfrak{U} \prec \mathfrak{V}$ (i.e., $\mathfrak{U} \subset \overline{\mathfrak{V}}$) are two facets of $\mathcal{B}$, then we say that $\mathfrak{U}$ is a **face** of $\mathfrak{V}$.
- **Chambers** of $\mathcal{B}$ are maximal facets, i.e., $C \cap \mathcal{B}$ for some k-chamber C of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.
- **Apartments** of $\mathcal{B}$ are of the form $A \cap \mathcal{B}$, where A runs through special k-apartments of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. We also have $\dim A = \mathrm{rank}_k(G^{\mathrm{ss}})$.

Remark 2.2.2.

(1) Any chamber of $\mathcal{B}$, hence the affine building $\mathcal{B}$, has dimension $\mathrm{rank}_k(G^{\mathrm{ss}})$. In particular, k-chambers are of equal dimension. Moreover, note that a k-chamber needs not be a chamber.

(2) Under these construction, there is a **bijective correspondence** (compare with (2.1))

$$\{\text{apartments of } \mathcal{B}\} \to \{\text{maximal } k\text{-split tori of } G\}.$$

2.2.2 Results on anisotropic groups

Lemma 2.2.3. *Let A be a special k-apartment of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.*

(1) *Let C be a k-chamber contained in A and let $x \in \mathcal{B}$ be a point. There exists a spacial k-apartment containing both C and x.*

(2) *In particular, any point of $\mathcal{B}$ lies in a special k-apartment of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.*

Proof. See [Pra20, Proposition 2.6]. □

Proposition 2.2.4.

(1) *Given distinct points $x \neq y \in \mathcal{B}$, there exists a spacial k-apartment containing $\{x, y\}$.*

(2) *Given any two k-facets of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$, there is a spacial k-apartment containing them.*

Proof.

(1) Let F be a k-facet of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ containing y. Let C be a maximal k-facet such that $F \prec C$. Thus C is a k-chamber (because it is maximal). Let $z \in C \cap \mathcal{B}$. According to Lemma 2.2.3(2), there is a special k-apartment A_0 containing z. Thus $C \subset A_0$ (by the same proof as Proposition 2.1.24). Then applying Lemma 2.2.3(1) to C and x yields a special k-apartment A contain both C and x. In particular, $\overline{C} \subset A$. Thus $F \subset \overline{C} \subset A$, and so $y \in A$ as well.

(2) Let F_1 and F_2 be two k-facets of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ and let C_i be respective k-chambers such that $F_i \prec C_i$. Suppose $z_i \in C_i \cap \mathcal{B}$ and let A be a special k-apartment containing z_i. Thus $C_i \subset \overline{C}_i \subset A$. In particular, $F_i \subset \overline{C}_i$ lies in A. □

The above results are useful in the study of anisotropic groups.

Proposition 2.2.5. *If G is k-anisotropic, then $\mathcal{B} = \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})^{\Gamma}$ is a singleton.*

Proof. Suppose that $\mathcal{B}$ contains two distinct points $x \neq y$. Applying Proposition 2.2.4(1) yields a special k-apartment A of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ containing $\{x, y\}$. Let $T \subset G$ be the special k-torus corresponding to A. Recall Hypothesis 2.1.4 that A is an affine space under $V(T_{\mathrm{nr}}) := \mathbf{X}_*(k_{\mathrm{nr}}, T) \otimes_{\mathbb{Z}} \mathbb{R}$. Since G is k-anisotropic, T is k-anisotropic as well. Thus

$$V(T_{\mathrm{nr}})^{\Gamma} = (\mathbf{X}_*(k_{\mathrm{nr}}, T) \otimes_{\mathbb{Z}} \mathbb{R})^{\Gamma} = \mathbf{X}_*(k, T) \otimes_{\mathbb{Z}} \mathbb{R} = 0$$

is trivial. Consequently, A^{Γ} is a singleton as well which contradicts to $x \neq y \in A^{\Gamma} \subset \mathcal{B}$. □

2.2.3 Unramified descent of buildings

In this subsection, we show that the Bruhat–Tits buildings descend from k_{nr} to k. More precisely, we show that the polysimplicial complex $\mathcal{B} := \mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})^{\Gamma}$ is an affine building. We may assume that G is semi-simple (Hypothesis 2.1.3). Hence $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}}) = \mathcal{B}(G/k_{\mathrm{nr}})$.

Theorem 2.2.6. *The polysimplicial complex $\mathcal{B} := \mathcal{B}(G/k_{\mathrm{nr}})^{\Gamma}$ is an affine building.*

- *Its apartments are $\mathfrak{A} := A \cap \mathcal{B}$ for special k-apartments of $\mathcal{B}(G/k_{\mathrm{nr}})$.*
- *Its chambers are $\mathfrak{C} := C \cap \mathcal{B}$ for k-chambers of $\mathcal{B}(G/k_{\mathrm{nr}})$.*
- *Its facets are $\mathfrak{F} := F \cap \mathcal{B}$ for k-facets of $\mathcal{B}(G/k_{\mathrm{nr}})$.*

In addition, the group $G(k)$ acts on $\mathcal{B}$ by polysimplicial isometries and apartments are affine spaces $\mathcal{B}(C_G(S)/k_{\mathrm{nr}})^{\Gamma}$ under $V(S) := \mathbf{X}_*(k, S) \otimes_{\mathbb{Z}} \mathbb{R}$ for some maximal k-split tori S of G.

Remark 2.2.7. For convenience, we recall what we need to show here.

(1) The polysimplicial complex $\mathcal{B}$ is a chamber complex. See Proposition 2.2.8 below.

(2) The chamber complex $\mathcal{B}$ is thick and any apartment in $\mathcal{B}$ is a thin chamber complex. See Proposition 2.2.9 below.

(3) Any two chambers belong to an apartment. This is already done in Proposition 2.2.4.

(4) Given two apartments $\mathcal{A}_1, \mathcal{A}_2$ and two facets $F_1, F_2 \in \mathcal{A}_1 \cap \mathcal{A}_2$, there exists a polysimplicial isomorphism $\mathcal{A}_1 \to \mathcal{A}_2$ fixing F_1 and F_2 pointwise. See Proposition 2.2.10 below.

(5) The last statement on the description $\mathcal{B}(C_G(S)/k_{\mathrm{nr}})^{\Gamma}$ will be omitted. See for instance [Pra20, Proposition 2.10] for a proof. Thus apartments are indeed affine spaces!

We first show that $\mathcal{B}$ is a chamber complex. To this end, it suffices to show the following

Proposition 2.2.8. *Let $\mathfrak{A}$ be an apartment of $\mathcal{B}$ and let $\mathfrak{C}, \mathfrak{C}'$ be two chambers in $\mathfrak{A}$. There is a gallery joining $\mathfrak{C}$ and $\mathfrak{C}'$ in $\mathfrak{A}$, i.e., there is a finite sequence*

$$\mathfrak{C} = \mathfrak{C}_0, \mathfrak{C}_1, \ldots, \mathfrak{C}_n = \mathfrak{C}'$$

of chambers in $\mathfrak{A}$ such that $\mathfrak{C}_i$ and $\mathfrak{C}_{i+1}$ share a facet of codimension 1 for each $0 \leq i \leq n-1$.

Proof. Let $\mathfrak{A}^{(2)}$ be the union of all facets in $\mathfrak{A}$ of codimension at least 2. Then $\mathfrak{A}^{(2)}$ is a closed subset of the affine space $\mathfrak{A}$ of codimension 2, so $\mathfrak{A} - \mathfrak{A}^{(2)}$ is arcwise connected for dimension reasons. Thus for any $x \in \mathfrak{C}$ and $x' \in \mathfrak{C}'$, there is a polyline in $\mathfrak{A} - \mathfrak{A}^{(2)}$ joining x and x'. Now the chambers in $\mathfrak{A}$ meeting this polyline form a gallery joining $\mathfrak{C}$ and $\mathfrak{C}'$. $\square$

A **panel** in a chamber complex is a facet of codimension 1.

Proposition 2.2.9. *The chamber complex $\mathfrak{B}$ is thick and any apartment $\mathfrak{A}$ of $\mathfrak{B}$ is thin.*

Proof. By definition, we need to show that:

(1) any panel in $\mathcal{B}$ is a face of at least three chambers;

(2) any panel in $\mathfrak{A}$ is a face of exactly two chambers in $\mathfrak{A}$.

Let $\mathfrak{F} := F \cap \mathcal{B}$ be a panel for some k-facet F of $\mathcal{B}(G/k_{\mathrm{nr}})$ (which is not a k-chamber for dimension reasons). Recall that any minimal pseudo-parabolic κ-subgroup of $\mathscr{G}^{\circ}_{F,\kappa}$ determines a k-chamber containing F. Thus proving both assertions needs to consider the number of minimal pseudo-parabolic κ-subgroups of $\mathscr{G}^{\circ}_{F,\kappa}$.

(1) We show that $\mathscr{G}^{\circ}_{F,\kappa}$ contains at least three distinct minimal pseudo-parabolic subgroups.

- If κ is finite, then pseudo-parabolic subgroups are parabolic. Note that any non-trivial irreducible projective κ-variety has at least three κ-point. Thus $\mathscr{G}^{\circ}_{F,\kappa}$ contains at least three parabolic subgroups by passing to quotients.
- If κ is infinite, then $\mathscr{G}^{\circ}_{F,\kappa}$ contains infinitely many minimal pseudo-parabolic subgroups.

Therefore, $\mathfrak{F}$ is at least a face of three distinct chambers.

(2) Similarly, we show that there exist exactly two minimal pseudo-parabolic subgroups. Suppose that $\mathfrak{F}$ is contained in an apartment $\mathfrak{A}$ of $\mathcal{B}$. Let S be the maximal k-split torus of G corresponding to $\mathfrak{A}$ and let $\mathscr{S}$ be the closed $\mathcal{O}$-split torus of $\mathscr{G}^{\circ}_{F} = \mathscr{G}^{\circ}_{\mathfrak{F}}$ with generic fibre S. Then there is a bijective correspondence between

$$\{\text{chambers of } \mathcal{B} \text{ contained in } \mathfrak{A} \text{ with } \mathfrak{F} \text{ a face}\}$$

and

$$\{\text{minimal pseudo-parabolic } \kappa\text{-subgroups of } \mathscr{G}^{\circ}_{F,\kappa}/\operatorname{rad}^u(\mathscr{G}^{\circ}_{F,\kappa}) \text{ containing } \overline{\mathscr{S}_{\kappa}}\}.$$

By Remark 2.1.21(4), the κ-rank of the derived subgroup of $\mathscr{G}^\circ_{F,\kappa}/\operatorname{rad}^u(\mathscr{G}^\circ_{F,\kappa})$ equals $\operatorname{codim}(\mathfrak{F},\mathcal{B})=1$. Therefore we conclude that $\mathscr{G}^\circ_{F,\kappa}/\operatorname{rad}^u(\mathscr{G}^\circ_{F,\kappa})$ has exactly two minimal pseudo-parabolic κ-subgroups containing $\overline{\mathscr{S}_\kappa}$. (For instance, see [Mil17, 25.14] on semi-simple K-groups of K-rank 1.) $\square$

Finally, we prove that for any chambers $\mathfrak{A}_1, \mathfrak{A}_2$ of $\mathcal{B}$, there exists a polysimplicial isomorphism $\mathfrak{A}_1 \to \mathfrak{A}_2$ fixing two given facets contained in $\mathfrak{A}_1 \cap \mathfrak{A}_2$.

Proposition 2.2.10. *For $i=1,2$, let A_i be special k-apartments of $\mathcal{B}(G/k_{\mathrm{nr}})$ with corresponding special k-torus T_i. Let $\mathfrak{A}_i := A_i \cap \mathcal{B}$. Let S be a k-split torus contained in $T_1 \cap T_2$. Let $\Omega \neq \varnothing$ be a Γ-stable bounded subset of $A_1 \cap A_2$. There exists an element $g \in \mathscr{G}^\circ_\Omega(\mathcal{O}) \subset G(k)$ that commutes with S and sends $\mathfrak{A}_1$ onto $\mathfrak{A}_2$.*

Proof. Let S_i be the maximal k-split torus of G contained in T_i. Let $\mathscr{T}_i$ be the closed $\mathcal{O}$-tori of $\mathscr{G}^\circ_\Omega$ with generic fibre T_i and let $\mathscr{S}_i$ be the maximal $\mathcal{O}$-split subtori. Thus the generic fibre of $\mathscr{S}_i$ is S_i. Let $\mathscr{S}$ be the closed $\mathcal{O}$-torus contained in $\mathscr{S}_1 \cap \mathscr{S}_2$ whose generic fibre is S. The construction may be visualized by the following diagrams

$$\begin{tikzcd}
 & S_i \arrow[rr] \arrow[dl] \arrow[ddl] & & \mathscr{S}_i \arrow[dl] \arrow[ddl] \\
T_i \arrow[rr] \arrow[d] & & \mathscr{T}_i \arrow[d] & \\
\operatorname{Spec} k \arrow[rr] & & \operatorname{Spec} \mathcal{O} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S \arrow[rr] \arrow[dl] \arrow[ddl] & & \mathscr{S} \arrow[dl] \arrow[ddl] \\
S_1 \cap S_2 \arrow[rr] \arrow[d] & & \mathscr{S}_1 \cap \mathscr{S}_2 \arrow[d] & \\
\operatorname{Spec} k \arrow[rr] & & \operatorname{Spec} \mathcal{O}. &
\end{tikzcd}$$

Let $\mathscr{M} := C_{\mathscr{G}^\circ_\Omega}(\mathscr{S})$ which is a smooth affine $\mathcal{O}$-subgroup scheme by [SGA3II, Exposé XI, Corollaire 5.3]. Fibres of $\mathscr{M}$ are connected since the centralizer of a torus in a connected smooth affine group scheme is connected.

Applying Lemma 2.1.18 to $\mathscr{M}$, we see that the special fibres $\mathscr{S}_{1,\kappa}$ and $\mathscr{S}_{2,\kappa}$ are maximal κ-split tori in $\mathscr{M}_\kappa$. Hence there exists $\overline{g} \in \mathscr{M}_\kappa(\kappa)$ that conjugates $\mathscr{S}_{1,\kappa}$ onto $\mathscr{S}_{2,\kappa}$. By Lemma 2.1.18(2), there exists $g \in \mathscr{M}(\mathcal{O}) \subset G(k)$ lying over $\overline{g}$ such that $g\mathscr{S}_1 g^{-1} = \mathscr{S}_2$. In particular, we obtain $gS_1g^{-1} = S_2$ by taking generic fibres. Hence

$$g \cdot A_1^\Gamma = g \cdot \mathcal{B}(C_G(S_1)/k_{\mathrm{nr}})^\Gamma = \mathcal{B}(C_G(S_2)/k_{\mathrm{nr}})^\Gamma = A_2^\Gamma.$$

Finally, note that since $g \in \mathscr{M}(\mathcal{O}) \subset \mathscr{G}^\circ_\Omega(\mathcal{O})$, it fixes Ω pointwise. $\square$

2.2.4 Conjugacy of subgroups

In this subsection, we collect several results on conjugate subgroups which will be used soon. Let $\mathfrak{G}$ be a connected smooth affine algebraic group over a field K. The following results tell us that maximal tori are conjugate. See [Mil17, Theorem 17.10 and Theorem 17.105] for proofs of the following result.

Theorem 2.2.11.

(1) *Maximal tori of* $\mathfrak{G}$ *are conjugate under* $\mathfrak{G}(K)$.

(2) *Maximal* K*-split tori of* $\mathfrak{G}$ *are conjugate under* $\mathfrak{G}(K)$.

We shall also need to consider parabolic subgroups. See [Mil17, Theorem 17.9 and Theorem 25.8] and [CGP15, Theorem C.2.5] respectively for proofs of the following theorem.

Theorem 2.2.12.

(1) *Suppose that* $\mathfrak{G}$ *is quasi-split. Borel subgroups of* $\mathfrak{G}$ *are conjugate under* $\mathfrak{G}(K)$.

(2) *Minimal parabolic subgroups of* $\mathfrak{G}$ *are conjugate under* $\mathfrak{G}(K)$.

(3) *Minimal pseudo-parabolic subgroups of* $\mathfrak{G}$ *are conjugate under* $\mathfrak{G}(K)$.

2.2.5 Transitive group actions

In this subsection, we still assume that G is **semi-simple** for simplicity.

Proposition 2.2.13. *Let* $\mathfrak{A}$ *be an apartment of* $\mathcal{B}$ *with corresponding maximal* k*-split torus* $S \subset G$*. Then the group* $N_G(S)(k)$ *acts transitively on the set of chambers of* $\mathfrak{A}$.

Proof. Let $\mathfrak{C} \neq \mathfrak{C}'$ be two chambers of $\mathfrak{A}$ and let $\mathfrak{C} = \mathfrak{C}_0, \mathfrak{C}_1, \ldots, \mathfrak{C}_n = \mathfrak{C}'$ be a gallery joining $\mathfrak{C}$ and $\mathfrak{C}'$. If there exists $g_i \in N_G(S)(k)$ such that $g_i \cdot \mathfrak{C}_i = \mathfrak{C}_{i+1}$ for $0 \leq i \leq n-1$, then $g := g_0 \cdots g_{n-1}$ is such that $g \cdot \mathfrak{C} = \mathfrak{C}'$. Hence it will be sufficient to show that if two chambers $\mathfrak{C} \neq \mathfrak{C}'$ in $\mathfrak{A}$ sharing a panel $\mathfrak{F}$, there is an element $g \in N_G(S)(k)$ such that $g \cdot \mathfrak{C} = \mathfrak{C}'$.

Suppose that $\mathfrak{C}, \mathfrak{C}'$ correspond to respective minimal pseudo-parabolic κ-subgroups $\overline{\mathscr{P}}, \overline{\mathscr{P}}'$ of $\mathscr{G}^{\circ}_{\mathfrak{F},\kappa}$. Let $\mathscr{S} \subset \mathscr{G}^{\circ}_{\mathfrak{F},\kappa}$ be the closed $\mathcal{O}$-split torus with generic fibre S. So $\mathscr{S}_\kappa \subset \mathscr{G}_\kappa$ is the maximal κ-split torus by construction. Moreover, since $\mathfrak{C}, \mathfrak{C}' \subset \mathfrak{A}$, we see that $\mathscr{S}_\kappa \subset \overline{\mathscr{P}}$ and $\mathscr{S}_\kappa \subset \overline{\mathscr{P}}'$ by Lemma 2.1.20.

By Theorem 2.2.11 and 2.2.12, there is an element $\overline{g} \in \mathscr{G}_\kappa(\kappa)$ which normalizes $\mathscr{S}_\kappa$ and conjugates $\overline{\mathscr{P}}$ onto $\overline{\mathscr{P}}'$. Now applying Lemma 2.1.18(iii) yields an element $g \in N_{\mathscr{G}}(\mathscr{S})(\mathcal{O})$ lying over $\overline{g}$. Note that g normalizes S (by taking generic fibres) and hence $g \in N_G(S)(k)$. It fixes $\mathfrak{F}$ pointwise and $g \cdot \mathfrak{F} = \mathfrak{F}'$. $\square$

Corollary 2.2.14. *The group* $G(k)$ *acts transitively on the set of ordered pairs consisting of an apartment* $\mathfrak{A}$ *of* $\mathcal{B}$ *and a chamber* $\mathfrak{C}$ *in* $\mathfrak{A}$.

Proof. Let $(\mathfrak{A}, \mathfrak{C})$ and $(\mathfrak{A}', \mathfrak{C}')$ be two such pairs. Since maximal k-split tori of G are conjugate to each other under $G(k)$, we conclude that $G(k)$ acts transitively on the set of apartments of $\mathcal{B}$. Take $g \in G(k)$ such that $\mathfrak{A}' = g \cdot \mathfrak{A}$.

Let S' be the maximal k-split torus corresponding to $\mathfrak{A}'$. Applying Proposition 2.2.13 to $\mathfrak{C}'$ and $g \cdot \mathfrak{C}$ in $\mathfrak{A}'$ yields an element $n \in N_G(S')(k)$ such that $\mathfrak{C}' = n \cdot (g \cdot \mathfrak{C})$. Therefore $(ng) \cdot (\mathfrak{A}, \mathfrak{C}) = (\mathfrak{A}', \mathfrak{C}')$. $\square$

2.2.6 Parahoric subgroups

Let G be a connected reductive group over k. For any point $x \in \mathcal{B}$, let $\mathscr{G}_x^\circ$ be the Bruhat–Tits parahoric $\mathcal{O}$-group scheme associated to x and let $\mathscr{G}_x^\circ(\mathcal{O})$ be the parahoric subgroup of $G(k)$ associated to x. We summarize some results for this setting.

(1) By construction of $\mathscr{G}_x^\circ$, its generic fibre is the connected reductive group G.

(2) Let $\mathfrak{F} := F \cap \mathcal{B}$ be a facet of $\mathcal{B}$ containing x, where F is a k-facet of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. We have $\mathscr{G}_x^\circ = \mathscr{G}_{\mathfrak{F}}^\circ = \mathscr{G}_F^\circ$ by Hypothesis 2.1.7(2).

(3) The subgroup $\mathscr{G}_x^\circ(\mathcal{O}) = \mathscr{G}_F^\circ(\mathcal{O}) = \mathscr{G}_F^\circ(\mathcal{O}_{\mathrm{nr}})^\Gamma = \mathscr{G}_F^\circ(\mathcal{O}_{\mathrm{nr}}) \cap G(k)$ of $G(k)$ fixes F pointwise.

Lemma 2.2.15.

(1) *Any parahoric subgroup of $G(k)$ is of the form P^Γ for some Γ-stable parahoric subgroup P of $G(k_{\mathrm{nr}})$.*

(2) *Any minimal parahoric subgroup of $G(k)$ is of the form $\mathscr{G}_{\mathfrak{C}}^\circ(\mathcal{O})$ for a chamber $\mathfrak{C}$ of $\mathcal{B}$.*

Proof.

(1) Let $\mathscr{G}_{\mathfrak{F}}^\circ(\mathcal{O})$ be a parahoric subgroup of $G(k)$. Then $\mathscr{G}_{\mathfrak{F}}^\circ(\mathcal{O}) = \mathscr{G}_F^\circ(\mathcal{O}_{\mathrm{nr}})^\Gamma = P^\Gamma$ with $P = \mathscr{G}_F^\circ(\mathcal{O}_{\mathrm{nr}})$, as desired.

(2) Let $\mathscr{G}_{\mathfrak{F}}^\circ(\mathcal{O})$ be a minimal parahoric subgroup of $G(k)$ and let F be the unique k-facet of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ containing $\mathfrak{F}$. Thus F is contained in some maximal k-facet C. It follows that $\mathscr{G}_{\mathfrak{C}}^\circ(\mathcal{O}) \subset \mathscr{G}_{\mathfrak{F}}^\circ(\mathcal{O})$, where $\mathfrak{C} = C \cap \mathcal{B}$. In particular, we must have $\mathfrak{F} = \mathfrak{C}$, i.e., $\mathfrak{F}$ is a chamber. □

Proposition 2.2.16. *The minimal parahoric subgroups of $G(k)$ are conjugate under $G^{\mathrm{ss}}(k)$.*

Proof. The minimal parahoric subgroups of $G(k)$ are the subgroups $\mathscr{G}_{\mathfrak{C}}^\circ(\mathcal{O})$ for chambers $\mathfrak{C}$ in $\mathcal{B}$. Corollary 2.2.14 implies that $G^{\mathrm{ss}}(k)$ acts transitively on the set of chambers of $\mathcal{B}$. □

Proposition 2.2.17. *Let G be a semi-simple simply connected group over k. Suppose that G is quasi-split over k_{nr}. If $P \subset G(k)$ is a parahoric subgroup, then $N_{G(k)}(P) = P$.*

Proof. Let $\mathfrak{F}$ be a facet of $\mathcal{B}$ and F the k-facet of $\mathcal{B}(G/k_{\mathrm{nr}})$ containing $\mathfrak{F}$. Then we have

$$\mathrm{Stab}_{G(k_{\mathrm{nr}})}(\mathfrak{F}) = \mathscr{G}_{\mathfrak{F}}^\circ(\mathcal{O}_{\mathrm{nr}}) = \mathscr{G}_F^\circ(\mathcal{O}_{\mathrm{nr}})$$

by Hypothesis 2.1.7. It follows that

$$\mathrm{Stab}_{G(k)}(\mathfrak{F}) = \mathscr{G}_{\mathfrak{F}}^\circ(\mathcal{O}).$$

Again by Hypothesis 2.1.7, we conclude that $\mathscr{G}_{\mathfrak{F}}^\circ = \mathscr{G}_{\mathfrak{F}} = \mathscr{G}_F$. Thus $\mathrm{Stab}_{G(k)}(\mathfrak{F})$ fixes $\mathfrak{F}$ and F pointwise.

Now suppose $P = \mathscr{G}_{\mathfrak{F}}^\circ(\mathcal{O}) = \mathrm{Stab}_{G(k)}(\mathfrak{F})$ for some facet $\mathfrak{F}$ of $\mathcal{B}$. Then $N_{G(k)}(P)$ also stabilizes $\mathfrak{F}$, and hence it coincides with P. □

Chapter 3

Application to Galois Cohomology

3.1 Preliminaries

3.1.1 Galois cohomology

Cohomology of profinite groups

Let Δ be a profinite group and let $\mathfrak{G}$ be an abstract Δ-group. We denote the Δ-action on $\mathfrak{G}$ by $(\sigma, g) \mapsto {}^{\sigma}g$.

Definition 3.1.1.

(1) A **cocycle** (or a crossed homomorphism) $c : \Delta \to \mathfrak{G}$ is a map such that

$$c_{\sigma\tau} = c_{\sigma} \cdot {}^{\sigma}c_{\tau}.$$

The set of cocycles is denoted by $Z^1(\Delta, \mathfrak{G})$.

(2) Two cocycles $c, z \in Z^1(\Delta, \mathfrak{G})$ are **cohomologous** if

$$z_{\sigma} = g^{-1} \cdot c_{\sigma} \cdot {}^{\sigma}g.$$

We simply write $c \sim z$ if they are cohomologous.

(3) We put

$$H^1(\Delta, \mathfrak{G}) := Z^1(\Delta, \mathfrak{G})/\sim .$$

Remark 3.1.2.

(1) Actually, we have the following description

$$H^1(\Delta, \mathfrak{G}) = \bigcup H^1(\Delta/U, \mathfrak{G}^U)$$

where U runs through all open normal subgroups of Δ. Moreover, the union is a directed union.

(2) If $\mathfrak{G}$ is a Δ-module, then $H^i(\Delta, \mathfrak{G}) = \bigcup H^i(\Delta/U, \mathfrak{G}^U)$ is a torsion abelian group for each $i \geq 1$. Here $H^i(\Delta/U, \mathfrak{G}^U)$ is the usual abelian cohomology group of finite groups.

Galois cohomology and inner twists

Let K be a field and let K_{sep} be a fixed separable closure of K. Then $\text{Gal}(K_{\text{sep}}|K)$ is a profinite group and $\mathfrak{G}(K_{\text{sep}})$ is a Galois module.

Definition 3.1.3. Let $\mathfrak{G}$ be an algebraic group over K. We put

$$H^1(K,\mathfrak{G}) := H^1(\text{Gal}(K_{\text{sep}}|K), \mathfrak{G}(K_{\text{sep}})).$$

Let $\mathfrak{G}$ be a smooth algebraic group over K. We have a homomorphism of $\text{Gal}(K_{\text{sep}}|K)$-groups

$$\mathfrak{G}(K_{\text{sep}}) \to \text{Aut}(\mathfrak{G}_{\text{sep}}),\ g \mapsto \text{Int}(g)$$

which induces a map of pointed sets

$$H^1(K,\mathfrak{G}) \to H^1(K, \text{Aut}(\mathfrak{G}_{\text{sep}})).$$

The image of $[c] \in H^1(K,\mathfrak{G})$ in $H^1(K, \text{Aut}(\mathfrak{G}_{\text{sep}}))$ defines a twist $\mathfrak{G}^c$, called the **inner Galois twist of $\mathfrak{G}$**. We shall also say that $\mathfrak{G}$ and $\mathfrak{G}^c$ have **the same inner type**.

The homomorphism of algebraic groups

$$\mathfrak{G} \to \text{Inn}(\mathfrak{G}),\ g \mapsto \text{Int}(g)$$

induces an isomorphism of algebraic groups

$$\mathfrak{G}^{\text{adj}} := \mathfrak{G}/Z(\mathfrak{G}) \simeq \text{Inn}(\mathfrak{G}).$$

Thus algebraic groups having the same inner type as $\mathfrak{G}$ are classified by $H^1(K,\mathfrak{G}^{\text{adj}})$.

Cohomological dimensions

Definition 3.1.4. Let K be a field.

(1) We say that K has **cohomological dimension $\leq d$**, denoted by $\text{cd}(K) \leq d$, if for any finite $\text{Gal}(K_{\text{sep}}|K)$-module A, we have $H^n(K, A) = 0$ for any $n \geq d + 1$.

(2) We say that K has **cohomological dimension d**, denoted by $\text{cd}(K) = d$, if

$$d = \max\{n \,|\, \text{there is a finite } \text{Gal}(K_{\text{sep}}|K)\text{-module } A \text{ such that } H^n(K, A) \neq 0\}.$$

(3) We say that K is a **C_1-field** if every homogenous polynomial $f \in K[t_1, \dots, t_n]$ of degree $d < n$ has a non-trivial zero.

For fields of cohomological dimension ≤ 1, we have a simpler description: $\text{cd}(K) \leq 1$ if and only if $\text{Br}(L) = 0$ for any algebraic extension $L|K$.

Example 3.1.5. We collect some well-known C_1-fields. In particular, they are of $\text{cd}(K) \leq 1$. See [GS17, §6.2] for more information.

(1) Every C_1-field has cohomological dimension ≤ 1.

(2) Finite extensions of C_1-fields are C_1-fields.

(3) (Chevalley). Finite fields are C_1-fields, hence they have cohomological dimension ≤ 1.

(4) (Tsen). The field $\mathbb{C}(t)$ is C_1, hence it has cohomological dimension ≤ 1.

(5) (Lang). Let K be a field with $\operatorname{char} K = 0$. The field $K(\!(t)\!)_{\mathrm{nr}}$ is a C_1-field.

(6) In particular, the field $K(\!(t)\!)$ is a C_1-field if K is algebraically closed.

Example 3.1.6.

(1) If K is separably closed, then $\mathrm{cd}(K) = 0$.

(2) If K is a finite field, then $\mathrm{cd}(K) = 1$.

(3) If K is a p-adic field or a totally imaginary number field, then $\mathrm{cd}(K) = 2$.

(4) If K is a real number field, then $\mathrm{cd}(K) = \infty$ since $\mathrm{cd}(\mathbb{R}) = \infty$.

3.1.2 A theorem of Steinberg–Borel–Springer

Let K be a field of $\mathrm{cd}(K) \leq 1$. Serre's conjecture I predicts $H^1(K, \mathfrak{G}) = 1$ for any connected affine algebraic group $\mathfrak{G}$ over K. It is proved first by Steinberg in 1962 for prefert fields and later by Borel–Springer in 1966 in general.

Theorem 3.1.7 (Steinberg, Borel–Springer)**.** *Let K be a field of $\mathrm{cd}(K) \leq 1$. Let $\mathfrak{G}$ be a connected affine algebraic group over K. Assume that either K is perfect or $\mathfrak{G}$ is reductive. Then $H^1(K, \mathfrak{G}) = 1$. In particular, Every connected affine algebraic group over K is quasi-split.*

Proof.

(1) We show that $H^1(K, \mathfrak{G}) = 1$.

- When K is perfect, see [Ser65, Chapter III, Section 2.3, Theorem 1′].
- When $\mathfrak{G}$ is reductive, see [BS68] and [Ser65, pp. 133, Remarks 1)].

(2) Let $\mathfrak{G}_0$ be a quasi-split K-group of the same inner type as $\mathfrak{G}$. Since $H^1(K, \mathfrak{G}_0^{\mathrm{ad}}) = 1$ by (1), there exists only one K-form of a given inner type. Hence $\mathfrak{G} = \mathfrak{G}_0$ is quasi-split. □

Proposition 3.1.8. *Let $\mathfrak{G}$ be a connected reductive group over k_{nr}. Let $k_{\mathrm{nr}}^{\wedge}$ be the completion of k_{nr}. Then the natural map of pointed sets below is bijective*

$$\rho : H^1(k_{\mathrm{nr}}, \mathfrak{G}) \to H^1(k_{\mathrm{nr}}^{\wedge}, \mathfrak{G}).$$

Sketch. We only sketch the idea here. See [GGMB14, Proposition 3.5.3(2)] for a proof.

Embed $\mathfrak{G}$ into $\mathbf{GL}_V$ and put $X = \mathbf{GL}_V/\mathfrak{G}$ (which is an algebraic space over k_{nr}). Then consider the following commutative diagram of pointed sets with exact rows

$$\begin{tikzcd} X(k_{\mathrm{nr}}) \arrow[r] \arrow[d] & H^1(k_{\mathrm{nr}}, \mathfrak{G}) \arrow[r] \arrow[d] & 1 \\ X(k_{\mathrm{nr}}^\wedge) \arrow[r] & H^1(k_{\mathrm{nr}}^\wedge, \mathfrak{G}) \arrow[r] & 1. \end{tikzcd}$$

Here we have used Hilbert's Theorem 90 to conclude $H^1(-, \mathbf{GL}_V) = 1$.

- Let E be a $\mathfrak{G}$-torsor over k_{nr} such that $E(k_{\mathrm{nr}}^\wedge) \neq \varnothing$. Then E is a separable algebraic space locally of finite type over k_{nr}. One obtains $E(k_{\mathrm{nr}}) \neq \varnothing$ by some sort of weak approximation [GGMB14, Proposition 3.5.2]. This shows the triviality of $\operatorname{Ker}\rho$. Hence ρ is injective by a twisting argument.

- Again by [GGMB14, Proposition 3.5.2], one concludes the surjectivity of $X(k_{\mathrm{nr}}) \to X(k_{\mathrm{nr}}^\wedge) \to H^1(k_{\mathrm{nr}}^\wedge, \mathfrak{G})$. Then the surjectivity of ρ follows by a diagram chase. $\square$

Corollary 3.1.9. *Let $\mathfrak{G}$ be a connected reductive group over k_{nr}. Suppose that κ is perfect with $\mathrm{cd}(\kappa) \leq 1$. Then $H^1(k_{\mathrm{nr}}, \mathfrak{G}) = 1$. In particular, every connected reductive k_{nr}-group is quasi-split.*

Proof. Let $k_{\mathrm{nr}}^\wedge$ be the completion of k_{nr} with respect to v. So v extends uniquely to $k_{\mathrm{nr}}^\wedge$ and the residue field of $k_{\mathrm{nr}}^\wedge$ coincides with that of k_{nr}. By assumption κ is perfect, thus $\kappa(k_{\mathrm{nr}}) = \kappa(k_{\mathrm{nr}}^\wedge) = \overline{\kappa}$ is the algebraic closure of κ. Consequently, $k_{\mathrm{nr}}^\wedge$ is a C_1-field by Lang's theorem. Hence $\mathrm{cd}(k_{\mathrm{nr}}^\wedge) \leq 1$. Applying Proposition 3.1.8 to $\mathfrak{G}$ implies that

$$H^1(k_{\mathrm{nr}}, \mathfrak{G}) \to H^1(k_{\mathrm{nr}}^\wedge, \mathfrak{G})$$

is bijective. Now Theorem 3.1.7 tells us that $H^1(k_{\mathrm{nr}}, \mathfrak{G}) = 1$, as desired. $\square$

3.1.3 Unramified Galois descent

Let $p : S' \to S$ be a morphism of schemes. Let $S'' = S' \times_S S'$ and let $S''' = S' \times_S S''$. We are interested in the essential image of the functor

$$p^* : \mathfrak{Sch}_S \to \mathfrak{Sch}_{S'},\ X \mapsto X \times_S S'.$$

Construction 3.1.10 (**Covering data**)**.** Consider the following Cartesian diagram

$$\begin{tikzcd} S'' \arrow[r, "p_1"] \arrow[d, "p_2"'] & S' \arrow[d, "p"] \\ S' \arrow[r, "p"] & S. \end{tikzcd}$$

Take any $X' \in \mathfrak{Sch}_{S'}$. An isomorphism $\varphi : p_1^* X' \to p_2^* X'$ is called a **covering datum** of X'. Thus we obtain a category $\mathfrak{Sch}_{S'}^{\mathrm{cov}}$ whose objects are pairs (X', φ) consisting of an S'-scheme X' and a covering datum φ of X'.

Note that if $X' = p^*X$ for some $X \in \mathfrak{Sch}_S$, then $p \circ p_1 = p \circ p_2$ gives us that

$$p_1^*(p^*X) = (p \circ p_1)^*X = (p \circ p_2)^*X = p_2^*(p^*X).$$

Thus we automatically obtain a covering datum of p^*X.

Construction 3.1.11 (**Descent data**)**.** Let $p_{ij} : S''' \to S''$ be respective projections for $1 \leq i < j \leq 3$. A necessary condition for $(X', \varphi) \in \mathfrak{Sch}_{S'}^{\mathrm{cov}}$ belonging to the essential image of $p^* : \mathfrak{Sch}_S \to \mathfrak{Sch}_{S'}$ is the commutativity of the diagram

$$\begin{tikzcd}
p_{12}^*p_1^*X' \arrow[r, "p_{12}^*\varphi"] \arrow[d, equal] & p_{12}^*p_2^*X' \arrow[r, equal] & p_{23}^*p_1^*X' \arrow[r, "p_{23}^*\varphi"] & p_{23}^*p_2^*X' \arrow[d, equal] \\
p_{13}^*p_1^*X' \arrow[rrr, "p_{13}^*\varphi"] & & & p_{13}^*p_2^*X'
\end{tikzcd}$$

The commutativity may be simply written as

$$p_{13}^*\varphi = p_{23}^*\varphi \circ p_{12}^*\varphi,$$

which is usually referred to as the **cocycle condition** for φ.

- A covering datum φ of X' satisfying the cocycle condition will be called a **descent datum** of X'. Similarly, we obtain a category $\mathfrak{Sch}_{S'}^{\mathrm{desc}}$.
- A descent datum is effective if the pair (X', φ) is isomorphic to p^*X for some $X \in \mathfrak{Sch}_S$, where p^*X is endowed with the canonical descent datum.

See [BLR90, §6.1, Theorem 6] for a proof of the following theorem.

Theorem 3.1.12. *Let $p : S' \to S$ be a fpqc morphism of* ***affine*** *schemes. A descent datum (X', φ) is effective if and only if there is an open quasi-affine covering $X' = \bigcup U'$ such that φ induces isomorphisms $p_1^*U' \simeq p_2^*U'$.*

Example 3.1.13 (Unramified Galois descent)**.** Let $p : \operatorname{Spec} \mathcal{O}_{\mathrm{nr}} \to \operatorname{Spec} \mathcal{O}$ (which is a Galois covering). Let $I = \mathscr{G}_C^\circ(\mathcal{O}_{\mathrm{nr}})$ be the Iwahori subgroup of $G(k_{\mathrm{nr}})$ for some chamber C. Then $H^1(\Gamma, I)$ classifies $\mathscr{G}_C^\circ(\mathcal{O}_{\mathrm{nr}})$-torsors with Γ-actions. But giving a Γ-action on such a torsor $\mathbf{X}$ is equivalent to give a descent datum on an $\mathcal{O}_{\mathrm{nr}}$-scheme $\widetilde{\mathscr{X}}$ with $\widetilde{\mathscr{X}}(\mathcal{O}_{\mathrm{nr}}) = \mathbf{X}$ by [BLR90, §6.2, Example B], so $\widetilde{\mathscr{X}}$ descends to a $\mathscr{G}_C^\circ$-torsor $\mathscr{X}$ over $\mathcal{O}$ by Theorem 3.1.12.

Note that $\mathscr{X}$ always admits an $\mathcal{O}_{\mathrm{nr}}$-point. Indeed, the $\mathcal{O}$-scheme $\mathscr{X}$ is smooth since it is a $\mathscr{G}_C^\circ$-torsor. On the other hand, $\mathscr{X} \times_{\mathcal{O}} \mathcal{O}_{\mathrm{nr}}$ always has a κ_{sep}-point, hence $\mathscr{X}(\mathcal{O}_{\mathrm{nr}}) \neq \varnothing$ by the Henselian assumption. Summing up, we have showed that $H^1(\Gamma, I) = H^1_{\text{ét}}(\mathcal{O}, \mathscr{G}_C^\circ)$.

3.2 Proof of main theorems

In this section, we assume throughout that κ is perfect and that $\mathrm{cd}(\kappa) \leq 1$. Thus the Bruhat–Tits theory is available for G over k_{nr}.

3.2.1 Preliminaries

Residually quasi-split groups

Corollary 3.2.1. *If κ is perfect and $\mathrm{cd}(\kappa) \leq 1$, then every k-chamber is a chamber of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$.*

Proof. Let C be a k-chamber of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ and let F be any k-facet contained in C. Then C corresponds to a minimal pseudo-parabolic κ-subgroup $\mathfrak{p}(F|C)$ of $\mathscr{G}^{\circ}_{F,\kappa}$. We need to show that $\mathfrak{p}(F|C)$ is a minimal pseudo-parabolic κ_{sep}-subgroup of $\mathscr{G}^{\circ}_{F,\kappa_{\mathrm{sep}}}$.

By assumption and Theorem 3.1.7(2), the group $\mathscr{G}^{\circ}_{F,\kappa}$ is quasi-split over κ. So minimal pseudo-parabolic κ-subgroups are Borel κ-subgroups. In particular, $\mathfrak{p}(F|C)$ is still a minimal pseudo-parabolic κ_{sep}-subgroup of $\mathscr{G}^{\circ}_{F,\kappa_{\mathrm{sep}}}$. So C is a chamber of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$. $\square$

Definition 3.2.2. Let G be a connected reductive group.

(1) We say that G is **residually quasi-split** if every k-chamber in $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ is a chamber.

(2) Suppose that G is residually quasi-split. The minimal parahoric subgroups of $G(k)$ are called the **Iwahori subgroups of $G(k)$**.

Example 3.2.3. If κ is perfect and $\mathrm{cd}(\kappa) \leq 1$, then every semi-simple k-group G is residually quasi-split by Corollary 3.2.1.

Results on anisotropic groups

Theorem 3.2.4. *Let G be an absolutely almost-simple simply connected group over k. If G is k-anisotropic, then G splits over k_{nr} and G is of type* $\mathbf{A}$.

To prove that G is of type $\mathbf{A}$, the idea is to prove that Γ acts transitively on a base of the affine Dynkin diagram of G_{nr} with respect to some maximal k_{nr}-split torus. Since we did not talk about the relation between roots systems and Bruhat–Tits theory, the following proof is not self-contained.

Proof. Let A be a special k-apartment of $\mathcal{B}(G/k_{\mathrm{nr}})$ and let T be the special k-torus of G corresponding to A. Let C be a k-chamber in A. By definition of k-chambers, $C \cap \mathcal{B} \neq \varnothing$. But G is k-anisotropic, so $\mathcal{B}$ is a singleton by Proposition 2.2.5. It follows that $C \cap \mathcal{B} = \mathcal{B}$.

Let $I = \mathscr{G}^{\circ}_{C}(\mathcal{O}_{\mathrm{nr}})$ be the Iwahori subgroup of $G(k_{\mathrm{nr}})$ determined by the chamber C of $\mathcal{B}(G/k_{\mathrm{nr}})$. Then I is Γ-stable. Because T is a special k-torus, T_{nr} is a maximal k_{nr}-split torus of G_{nr}. Consider the affine root system $\Phi^{\mathrm{aff}}(G_{\mathrm{nr}}, T_{\mathrm{nr}})$ over k_{nr} and let Δ be the base of $\Phi^{\mathrm{aff}}(G_{\mathrm{nr}}, T_{\mathrm{nr}})$ determined by the chamber C. According to the description of irreducible affine root systems, it suffices to show that Γ acts transitively on Δ.

By construction, Δ is Γ-stable[1] and there is a natural Γ-equivariant bijective map $\mathscr{V}_C \to \Delta$. By Bruhat–Tits theory, Γ acts transitively on $\mathscr{V}_C$, and hence it acts transitively on Δ.

Therefore G_{nr} is k_{nr}-split and that G is of type $\mathbf{A}$. Otherwise, the action of the automorphism group of the Dynkin diagram of Δ cannot be transitive on Δ. $\square$

[1] The Galois group Γ acts naturally on the affine root system $\Phi^{\mathrm{aff}}(G_{\mathrm{nr}}, T_{\mathrm{nr}})$.

Now we show that $H^1(k,G)=1$. We first observe that $H^1(k,G)=H^1(\Gamma,G(k_{\mathrm{nr}}))$. Then we show that $H^1(\Gamma,\mathscr{G}_C^\circ(\mathcal{O}_{\mathrm{nr}}))\to H^1(\Gamma,G(k_{\mathrm{nr}}))$ is surjective for some Iwahori subgroup. Subsequently, Galois descent tells us $H^1(\Gamma,I)=H^1_{\text{ét}}(\mathcal{O},\mathscr{G}_C^\circ)$. Thus it suffices to show that any $\mathscr{G}_C^\circ$-torsor $\mathscr{X}$ is trivial, i.e., $\mathscr{X}(\mathcal{O})\neq\varnothing$. This is guaranteed by the Henselian assumption and Steinberg's theorem.

Theorem 3.2.5. *Let G be a semi-simple simply connected group over k. Then $H^1(k,G)=1$.*

Proof. By Corollary 3.1.9, we have $H^1(k_{\mathrm{nr}},G)=1$. Thus

$$H^1(k,G)=H^1(\Gamma,G(k_{\mathrm{nr}}))=Z^1(\Gamma,G(k_{\mathrm{nr}}))/\sim,$$

where the first equality holds because $H^1(k,G)$ is a directed union. Take any $c:\Gamma\to G(k_{\mathrm{nr}})$, $\sigma\mapsto c_\sigma$ in $Z^1(\Gamma,G(k_{\mathrm{nr}}))$. Since $G_{\mathrm{nr}}\simeq({}_cG)_{\mathrm{nr}}$, we may identify $G(k_{\mathrm{nr}})$ with ${}_cG(k_{\mathrm{nr}})$ as abstract groups.[2]

Let $I=\mathscr{G}_C^\circ(\mathcal{O}_{\mathrm{nr}})$ be as above. Then I is also an Iwahori subgroup of ${}_cG(k_{\mathrm{nr}})$. On the other hand, ${}_cG$ is a residually quasi-split semi-simple group over k, ${}_cG(k_{\mathrm{nr}})$ contains an Iwahori subgroup which is stable under the twisted Γ-action. But Iwahori subgroups of ${}_cG(k_{\mathrm{nr}})$ are conjugate by Proposition 2.2.16, so there exists $g\in{}_cG(k_{\mathrm{nr}})=G(k_{\mathrm{nr}})$ such that gIg^{-1} is stable under the twisted Γ-action, i.e.

$$c_\sigma\cdot{}^\sigma(gIg^{-1})\cdot c_\sigma^{-1}=gIg^{-1}$$

for any $\sigma\in\Gamma$. In particular, the cocycle $z_\sigma:=g^{-1}\cdot c_\sigma\cdot{}^\sigma g\in G(k_{\mathrm{nr}})$ normalizes I. According to Proposition 2.2.17, the normalizer of I is I itself, so we conclude that $z\in Z^1(\Gamma,I)\subset Z^1(\Gamma,G(k_{\mathrm{nr}}))$ is cohomologous to c. Therefore it will be sufficient to show $H^1(\Gamma,I)=1$.

By unramified Galois descent Example 3.1.13, $H^1(\Gamma,I)=H^1(\Gamma,\mathscr{G}_C^\circ(\mathcal{O}_{\mathrm{nr}}))$ equals to $H^1_{\text{ét}}(\mathcal{O},\mathscr{G}_C^\circ)$. Consequently, it suffices to show $\mathscr{X}(\mathcal{O})\neq\varnothing$ for any $\mathscr{G}_C^\circ$-torsor over $\mathcal{O}$, i.e., any such torsor is trivial. By $\mathcal{O}$-smoothness of $\mathscr{X}$ and the Henselian assumption on $\mathcal{O}$, we reduce to prove that the special fibre of $\mathscr{X}$ has a κ-point. But the isomorphism class of the special fibre as a torsor over κ is determined by an element of the set $H^1(\Gamma,\mathscr{G}_C^\circ(\overline{\kappa}))$. According to Steinberg's theorem, this set is trivial. (Recall that κ is perfect and $\mathrm{cd}(\kappa)\leq 1$.) □

3.3 Conjugacy of special tori

Recall that κ is a perfect field such that $\mathrm{cd}(\kappa)\leq 1$. Thus every k-chamber is a chamber.

Lemma 3.3.1. *For $i=1,2$, let A_i be special k-apartments of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ with corresponding special k-torus T_i. Let $\mathfrak{A}_i:=A_i\cap\mathcal{B}$. Let S be a k-split torus contained in $T_1\cap T_2$. Let $\Omega\neq\varnothing$ be a Γ-stable bounded subset of $A_1\cap A_2$. There exists an element $g\in\mathscr{G}_\Omega^\circ(\mathcal{O})\subset G^{\mathrm{ss}}(k)$ commuting with S such that $T_2=gT_1g^{-1}$.*

Proof. This is a consequence of Proposition 2.2.10. □

[2] We emphasize that $G(k_{\mathrm{nr}})$ and ${}_cG(k_{\mathrm{nr}})$ are **not** identified as Γ-groups!

Theorem 3.3.2. *Let G be a connected reductive group over k. Let T_1 and T_2 be special k-tori of G. Then $T_2 = gT_1g^{-1}$ for some $g \in G^{\mathrm{ss}}(k)$.*

Proof. For $i = 1, 2$, let A_i be the corresponding special k-apartment of T_i.

- Suppose $A_1 \cap A_2 \neq \varnothing$. This follows from Lemma 3.3.1.
- Suppose $A_1 \cap A_2 = \varnothing$. Fix respective k-chambers C_i in A_i for $i = 1, 2$. Let A be a special k-apartment of $\mathcal{B}(G^{\mathrm{ss}}/k_{\mathrm{nr}})$ containing C_i for $i = 1, 2$. Let $T \subset G$ be the special k-torus corresponding to A. Applying Lemma 3.3.1 to the pair $\{A, A_i\}$ for $i = 1, 2$, we see that T is conjugate to T_i under $G^{\mathrm{ss}}(k)$. In particular, T_1 and T_2 are conjugate under $G^{\mathrm{ss}}(k)$ as well. □

Applying Theorem 3.3.2 to $C_G(S)$ yields the following corollary.

Corollary 3.3.3. *Let $S \subset G$ be a maximal k-split torus. Let $C_G(S)$ be the centralizer of S in G. Then any two special k-tori of $C_G(S)$ are conjugate under some $g \in (\mathscr{D}C_G(S))(k)$.*

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