On Computation of Double Dirichlet Series of Elliptic Curves over Finite Fields

Wei-Chen Yao
Department of Mathematics
University of Taipei

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Joint Work with Ting-Fang Lee

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Notation:

$\mathbb{F}_q$	:	finite field of odd characteristic.
C	:	smooth projective curve over $\mathbb{F}_q$.
K	:	function field of C.
$\mathrm{Div}(C)$	:	the group of $\mathbb{F}_q$-rational divisors on C.
$\mathrm{Div}_0(C)$	:	the divisors of degree 0.
$\mathrm{Div}_P(C)$	:	the subgroup of $\mathrm{Div}_0(C)$ consisting of the principal divisors.
$\mathrm{Pic}(C)$	:	$\mathrm{Div}(C)/\mathrm{Div}_P(C)$, the Picard group.
$\mathrm{Pic}_0(C)$	:	$\mathrm{Div}_0(C)/\mathrm{Div}_P(C)$.
$\mathrm{Div}(T)$	:	the subgroup of divisors whose support is contained in T where T is a finite or cofinite set of places of K.

Let $S \subseteq C$ be a finite set of places of K such that $\mathrm{Div}(S)$ represents all classes in $\mathrm{Pic}(C)$, and let

$$F = \sum_{v \in S} n_v v$$

be an effective divisor with $n_v \geq 1$ for all $v \in S$. We define

$$\mathrm{Div}_{P,F}(C - S) = \{(f) : f \in K^{\times}, \mathrm{ord}_v(f - 1) \geq n_v \text{ for all } v \in S\}.$$

Furthermore, define $\mathrm{Pic}_F(C) = \mathrm{Div}(C - S)/\,\mathrm{Div}_{P,F}(C - S)$ to be the ray class group modulo F and

$\mathrm{Pic}^0_F(C) = \mathrm{Div}_0(C - S)/\,\mathrm{Div}_{P,F}(C - S)$ to be the degree zero subgroup.

For $m \in K^\times$ and $D \in \mathrm{Div}(C)$, let

$$\left(\frac{m}{D}\right) = \prod_{v, \mathrm{ord}_v(m)=0} \left(\frac{m}{v}\right)^{\mathrm{ord}_v(D)},$$

where the right hand side is a product of Legendre symbols. Let χ_m be the character defined by

$$\chi_m(D) = \left(\frac{m}{D}\right)$$

if D is disjoint from (m), and $\chi_m((n)) = 1$ if $n \equiv 1$ modulo the square-free part of (m).

Fisher and Friedberg define a character χ_D on $\mathrm{Div}(C-S)$ as following:

Let $\mathcal{E}$ be a set of effective divisors of cosets representatives for $\mathrm{Div}(C-S)/[2\,\mathrm{Div}(C-S)+Div_{P,F}(C-S)] \simeq \mathrm{Pic}_F(C)\otimes\mathbb{Z}/2\mathbb{Z}$. For each $E\in\mathcal{E}$, we choose $m_E\in K^\times$ such that $E-(m_E)\in\mathrm{Div}(S)$.

If $D\in\mathrm{Div}(C-S)$, we can write $D=E+(m)+2G$ with $E\in\mathcal{E}$, $m\equiv 1 \mod F$, and $G\in\mathrm{Div}(C-S)$. We then define

$$\chi_D=\chi_{mm_E}.$$

Furthermore, we assume that $0\in\mathcal{E}$ and $m_0=1$, so that $\chi_{(m)}=\chi_m$ if $m\equiv 1 \mod F$.

The quadratic character χ_D is well-defined. It depends on the choice of the representatives $\mathcal{E} \subset \mathrm{Div}(C - S)$ and m_E, but not the choice of m. We also can choose $\mathcal{E}$ such that

$$\chi_{D+D'} = \chi_D \cdot \chi_{D'}.$$

If D is an effective divisor, let S_D be the support of the conductor of χ_D, and D' be its square-free part. Consistent with class field theory, we may check that $\chi_D(D_1) = \chi_{D'}(D_1)$ whenever both are defined. This allows us to extend χ_D to a character of $\mathrm{Div}(C - S - S_D)$.

Let ρ be a character of $\mathrm{Div}(C - S)$, and F_ρ be its conductor. We define the partial L-series

$$L(s, \rho; C - S) := \prod_{v \notin S} (1 - \rho(v)|v|^{-s})^{-1}$$

where $|v|$ is the norm of v and S is the support of F_ρ. Then the Euler product is also a sum over effective divisors:

$$L(s, \rho; C - S) = \sum_{0 \leq D \in \mathrm{Div}(C-S)} \rho(D)|D|^{-s}$$

for $\mathrm{Re}(s) > 1$. These L-series have an analytic continuation and functional equation.

Choose a divisor $B_1 \in \mathrm{Div}(C - S)$ of degree one. We define

$$X_F(C) = \mathrm{Div}(C - S)/(\langle 2B_1 \rangle + \mathrm{Div}_{P,F}(C - S)).$$

Let F be the fixed conductor. Let ρ be a character of $X_F(C)$, and μ be the Möbius function on $\mathrm{Div}(C)$. Given an effective divisor $D \in \mathrm{Div}(C - S)$ and let S_D denote the support of the conductor of χ_D. Define the finite sum

$$a(s, \rho, D) = \sum_{\substack{0 \le d_1 \in \mathrm{Div}(C - S - S_D) \\ 0 \le d_2 \in \mathrm{Div}(C - S) \\ 2(d_1 + d_2) \le D}} \mu(d_1)\chi_D(d_1)\rho(d_1 + 2d_2)|d_1|^{-s}|d_2|^{1-2s}.$$

Write $D = D_0 + 2D_1$ with D_0 square-free. This correction factor has a functional equation

$$a(s, \rho, D) = \rho(2D_1)|D_1|^{1-2s} a(1 - s, \rho^{-1}, D).$$

Define

$$\mathbf{L}(s,\rho,D) := L(s,\rho\chi_D; C - S - S_D)a(s,\rho,D).$$

Given ρ_1 and ρ_2 are characters on $X_F(C)$. We define the double Dirichlet series

$$Z(s,w;\rho_1,\rho_2) := \sum_{0 \le D \in \mathrm{Div}(C-S)} \rho_2(D)\mathbf{L}(s,\rho_1,D)|D|^{-w}$$

which converges absolutely in the tube domain $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(w) > 1$.

Since the definition of the characters χ_D depends on the choice of the representatives $\mathcal{E} \subset \mathrm{Div}(C - S)$, and the element $m_E, E \subset \mathcal{E}$. The twisted L-functions also depends on these choices. Moreover, the definition also depends on the choice the degree-one divisor $B_1 \in \mathrm{Div}(C - S)$. However, varying over ρ_1, ρ_2 gives a finite-dimensional vector space, which depends only on F.

Theorem (Fisher and Frideberg)

Let $V(F)$ be the span of $Z(s, w; \rho_1, \rho_2)$ for $\rho_1, \rho_2 \in \hat{X}_F(C)$. Then $V(F)$ is a finite-dimensional vector space of functions, which is independent of the choices of the representatives $\mathcal{E}$ and the element $m_E, E \in \mathcal{E}$. If in addition the exponent of the group $Pic_{F,0}$ is even, then the vector space is independent of the choice of the degree-one divisor B_1.

Remark. Let $R(F)$ be the vector space of functions on $X_F(C)$ with values in $\mathbb{C}$. Since $X_F(C)$ is a finite abelian group, its characters form a basis of $R(F)$. If $\sigma_j \in R(F), j = 1, 2$ and $\sigma_j = \sum c_{i,j}\rho_i$ with the ρ_i characters of $X(F)$ and $c_{i,j} \in \mathbb{C}$, then define

$$Z(s, w; \sigma_1, \sigma_2) = \sum_{i_1, i_2} c_{i_1,1} c_{i_2,2} Z(s, w; \rho_{i_1}, \rho_{i_2}).$$

In particular, for $x \in X_F(C)$, let δ_x denote the function on $\mathrm{Div}(C - S)$ given by $\delta_x(D) = 1$ if the class of D in $X_F(C)$ is x, $\delta_x(D) = 0$ otherwise. Then the δ_x for $x \in X(F, n)$, give another basis of $R(F)$, and $Z(s, w; \rho, \delta_x)$ is a partial Dirichlet series, the sum over all effective divisors of the form $D = E + (m) + 2jB_1$ with $m \equiv 1 \pmod F$ and j an integer, where E is a fixed representative of x.

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The first functional equation of $Z(s, w; \rho_1, \rho_2)$ can be derived by making use of the functional equation of $\mathbf{L}(s, \rho, D)$. We state it in the following:

Theorem (First Functional Equation, FF)

Let $x \in X_F(C)$ and let δ_x denote the characteristic function on $\mathrm{Div}(C - S)$*: $\delta_x(D) = 1$ if D is in the same coset with x, and $\delta_x(D) = 0$ otherwise. Let $E \in \mathcal{E}$ represent the coset of x in $X_F(C)$. Given a character ρ of $X_F(C)$, then*

$$Z(s, w; \rho, \delta_x) = \frac{\mathbf{L}(s, \rho, E)}{\mathbf{L}(1 - s, \rho^{-1}, E)} \times \rho(x - E)|E|^{s-1/2} Z(1 - s, s + w - \frac{1}{2}; \rho^{-1}, \delta_x)$$

on the tube domain
$\mathrm{Re}(w) > \max\{1, 3/2 - \mathrm{Re}(s), 3/2 - 1/2\,\mathrm{Re}(s)\}$.

The second functional equation sends $(s, w) \mapsto (w, s)$. We use the reciprocity law to analyze the relation between $\chi_{D_2}(D_1)$ and $\chi_{D_1}(D_2)$.

Lemma

Suppose $D, D' \in \mathrm{Div}(C - S)$ have disjoint support. Let

$$\alpha(D, D') := \chi_D(D')/\chi_{D'}(D).$$

Then $\alpha(D, D')$ depends only on the images of D and D' in $X_F(C)$.

The second functional equation is described as following:

Theorem (Second Functional Equation, FF)

Let η_1 and η_2 be functions on $X_F(C)$. For $E \in \mathcal{E}$, let δ_E denote the characteristic function $\delta_E(D) = 1$ if D and E represent the same class in $X_F(C)$, $\delta_E(D) = 0$ otherwise. Assuming that $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(w) > 1$, then

$$Z(s, w; \rho_1, \rho_2) = \sum_{E_1, E_2 \in \mathcal{E}} \alpha(E_2, E_1) Z(w, s; \delta_{E_2}\rho_2, \delta_{E_1}\rho_1).$$

In particular, for E_1, $E_2 \in \mathcal{E}$

$$Z(s, w; \delta_{E_1}, \delta_{E_2}) = \alpha(E_1, E_2) Z(w, s; \delta_{E_2}, \delta_{E_1}).$$

Since $(1-q^{2(1-s)})Z(s,w;\rho_1,\rho_2)$ converges absolutely and uniformly on compact in the tube domain T_1 defined by the condition $\mathrm{Re}(w) > \max\{1, \frac{3}{2} - \mathrm{Re}(s), \frac{3}{2} - \frac{1}{2}\mathrm{Re}(s)\}$. Let T_2 be the image of T_1 under the involution $(s,w) \mapsto (w,s)$, and T_3 be the image of T_2 under the involution $(s,w) \mapsto (1-s, s+w-\frac{1}{2})$.

Let $\Phi(s,w) := (1-q^{2(1-s)})(1-q^{2(1-w)})(1-q^{2(3/2-s-w)})$. Then $\Phi(s,w)Z(s,w;\rho_1,\rho_2)$ represents an analytic function on $T_1 \cup T_2 \cup T_3$. The convexity principle for analytic functions on tube domain gives the analytic continuation of $\Phi(s,w)Z(s,w;\rho_1,\rho_2)$ to all of $\mathbb{C}^2$.

Theorem (Fisher and Friedberg)

$Z(s,w;\rho_1,\rho_2)$ has meromorphic continuation to all of $\mathbb{C}^2$ and is a rational function in q^{-s} and q^{-w}. Moreover, $\Phi(s,w)Z(s,w;\rho_1,\rho_2)$ is a polynomial of degree at most $2\deg F + 4g$ in each of q^{-s} and q^{-w}.

Double Dirichlet Series for Elliptic Curves

Let C be an elliptic curve over $\mathbb{F}_q$ with q odd and K be the function field of C.

In the elliptic curve case, $C(\mathbb{F}_q) \longrightarrow \mathrm{Pic}^0(C)$ is an isomorphism of abelian groups where $C(\mathbb{F}_q)$ denotes the set of rational points. We choose $S = \{P_0, P_1, ..., P_{h-1}\}$ to be the finite set of places that each corresponds to a rational point on C where P_0 correspond to the infinity place. Define $F = \sum_{i=0}^{h-1} P_i$.

To formulate all functional equations, we need to know the structure of the quotient group $X_F(C)$. The first step is to study the ray class group $\mathrm{Pic}^0_F(C)$.

Suggested by computation, we have the following conjecture.

Conjecure

Suppose $\mathrm{Pic}^0(C) \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$ *for some positive integer* $n_1, \ldots, n_r$*. Then*

$$\mathrm{Pic}^0_F(C) \cong \Big(\bigoplus_{i=1}^{h-1-r} \mathbb{Z}/(q-1)\mathbb{Z}\Big) \oplus \Big(\bigoplus_{j=1}^{r} \mathbb{Z}/n_j(q-1)\mathbb{Z}\Big).$$

Although we can not prove the Conjecture, we can prove the following result.

Lemma

$\mathrm{Div}_p(C-S)/\,\mathrm{Div}_{P,F}(C-S) \simeq \mathbb{Z}/(q-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(q-1)\mathbb{Z}$ *where there are* $h-1$ *copies.*

The above conjecture gives the structure of the quotient group

$$\begin{aligned} X_F(C) &\cong (\mathrm{Pic}_F^0(C) \oplus \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \\ &\cong \mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

is a direct sum of h copies of $\mathbb{Z}/2\mathbb{Z}$. Next, we will determine a set of effective divisors which represents a coset of representatives of $X_F(C)$.
Let $C : y^2 = f(x)$ be an elliptic curve over $\mathbb{F}_q$ and let

$$\{P_\infty = P_0, P_1, P_2, \cdots, P_{h-1}\}$$

be the set of all degree one places on C. Find an element $\gamma \in \mathbb{F}_q$ such that $f(x) - \gamma^2$ is irreducible over $\mathbb{F}_q$. Then we can write $(y - \gamma) = Q_0 - 3P_\infty$ where Q_0 is a place of degree 3. Then we put

$$E_0 = Q_0 \quad \text{and} \quad m_0 := m_{E_0} = y - \gamma.$$

Let $P_i \neq P_\infty$ be a degree one place and let (α_i, β_i) be the rational point of C corresponding to P_i. We find a number $k \in F_q$ such that (α_i, β_i) is the only solution of the system

$$\begin{cases} y - \beta_i = k(x - \alpha_i) \\ y^2 = f(x) \end{cases} \qquad \text{in} \quad \mathbb{F}_q \times \mathbb{F}_q.$$

Hence

$$(y - k(x - \alpha_i) + \beta_i) = P_i + Q_i - 3P_\infty$$

where Q_i is a place of degree 2. Then we put

$$E_i = Q_i \quad \text{and} \quad m_i := m_{E_i} = y - k(x - \alpha_i) + \beta_i$$

for $i = 1, \cdots, h - 1$.

We define

$$\mathcal{E} := \{\sum_{i=0}^{h-1} \epsilon_i E_i | \epsilon_i = 0 \text{ or } 1\}$$

to be a set of coset representatives of the quotient group $X_F(C)$.

Definition

The double Dirichlet series of C is

$$Z(s,w;\mathrm{id},\mathrm{id}) = \sum_{0\le D\in \mathrm{Div}(C-S)} \mathbf{L}(s,\mathrm{id},D)|D|^{-w}.$$

By the rationality theorem, $Z(s,w;\mathrm{id},\mathrm{id})$ is a rational function with denominator $(1-q^2t^2)(1-q^2u^2)(1-q^3t^2u^2)$ and

$$(1-q^2t^2)(1-q^2u^2)(1-q^3t^2u^2)Z(s,w;\mathrm{id},\mathrm{id})$$

is a polynomial in t and u which the degree in t and u are less than $2h+4$ where $t=q^{-s}$ and $u=q^{-w}$.

By computing the Gauss sum and applying the functional equation of L-function, one can get the following lemma.

Lemma

Let ρ be a character of $X_F(C)$, $D = D_0 + 2D_1 \in \mathrm{Div}(C - S)$ with D_0, $D_1 \geq 0$ and D_0 square-free, $F_{\rho,D}$ the conductor of $\rho\chi_D$, and $S_{\rho,D}$ the support of $F_{\rho,D}$. Then

$$\frac{\mathbf{L}(s, \rho, D)}{\mathbf{L}(1-s, \rho^{-1}, D)} = \rho(2D_1)(q^{\deg(F_{\rho,D}+2D_1)})^{1/2-s} \times \prod_{v \in S - S_{\rho,D}} \frac{1 - \rho\chi_D(v)|v|^{-s}}{1 - \rho^{-1}\chi_D(v)|v|^{s-1}}.$$

In order to get more information of the second functional equation of double Dirichlet series, we will show several properties of the function α which is defined in last section. First of all, If D and D' are not disjoint, then we choose D_1 disjoint from D' in the same class in $X_F(C)$ with D, and define $\alpha(D, D') = \alpha(D_1, D')$.

Lemma

1. $\alpha(E_0, E_0) = \left(\frac{-1}{q}\right) = (-1)^{\frac{q-1}{2}}$
2. $\alpha(E_i, E_i) = \left(\frac{-1}{q}\right) = (-1)^{\frac{q-1}{2}}$ *for* $i = 1, \cdots, h-1$.
3. $\alpha(E_i, E_0) = \left(\frac{-1}{p_\infty}\right)\left(\frac{m_0}{P_i}\right) = (-1)^{\frac{q-1}{2}}\left(\frac{m_0}{P_i}\right)$ *for* $i = 1, \cdots h-1$.
4. $\alpha(E_i, E_j) = \left(\frac{-1}{P_\infty}\right)\left(\frac{m_i}{P_j}\right)^{-1}\left(\frac{m_j}{P_i}\right)$ *for* $i, j = 1, \cdots, h-1$ *and* $i \neq j$.

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4. $\alpha(E_i, E_j) = \left(\frac{-1}{P_\infty}\right)\left(\frac{m_i}{P_j}\right)^{-1}\left(\frac{m_j}{P_i}\right)$ *for* $i, j = 1, \cdots, h-1$ *and* $i \neq j$.

By the definition of partial L functions, $L(s, \mathrm{id}; C - S) = \zeta_{K,S}(s)$ where $\zeta_{K,S}(s)$ is the zeta function with the Euler factors in S is removed. Hence, we have the following Lemma.

Lemma

$$L(s, \mathrm{id}; C - S) = \frac{(1 - q^{-s})^{h-1}(1 + (h - 1 - q)q^{-s} + q^{1-2s})}{1 - q^{1-s}}.$$

Example

Let K be the function field of the curve $C: y^2 = x^3 + 2x$ over $\mathbb{F}_5$ (i.e. $q = 5$). The class number of K is 2. We can choose $S = \{P_\infty, P\}$ where P_∞ corresponds to the infinity place and P corresponds to $(0,0)$. Let $F = P_\infty + P$. The ray class group $\mathrm{Pic}_F(C) \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}$ gives $X(F) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Let $E_0 = 3 \cdot P_\infty + (y+1)$ and $E_1 = 3P_\infty - P + (y)$, then

$$\mathcal{E} = \{0, E_0, E_1, E_0 + E_1\}$$

is a set of coset representatives of $X_F(C)$. The quadratic characters attached to each divisor in $\mathcal{E}$ are:

$$\chi_0 = \mathrm{id}, \ \chi_{E_0} = \left(\frac{y+1}{\cdot}\right), \ \chi_{E_1} = \left(\frac{y}{\cdot}\right), \ \chi_{E_0+E_1} = \left(\frac{y(y+1)}{\cdot}\right).$$

By the arguments in last section, the double Dirichlet series of C is

$$Z(s,w;\mathrm{id},\mathrm{id}) = \frac{f(t,u)}{(1-q^2t^2)(1-q^2u^2)(1-q^3t^2u^2)}$$

where $t = q^{-s}$, $u = q^{-w}$ and $f(t,u)$ is a polynomial in t, u with degree ≤ 8. Let

$$Z(s,w;\mathrm{id},\delta_0) = \frac{f_0(t,u)}{(1-q^2t^2)(1-q^2u^2)(1-q^3t^2u^2)}$$
$$(:= \sum_{0\leq D\in \mathrm{Div}_{P,F}(C-S)+2\,\mathrm{Div}(C-S)} L(s,\chi_D; C-S-S_D)a(s,\mathrm{id},D)|D|^{-w})$$

$$Z(s,w;\mathrm{id},\delta_{E_0}) = \frac{f_{E_0}(t,u)}{(1-q^2t^2)(1-q^2u^2)(1-q^3t^2u^2)}$$
$$(:= \sum_{0\leq D\in E_0+\mathrm{Div}_{P,F}(C-S)+2\,\mathrm{Div}(C-S)} L(s,\chi_D; C-S-S_D)a(s,\mathrm{id},D)|D|^{-w})$$

$$Z(s,w;\mathrm{id},\delta_{E_1}) = \frac{f_{E_1}(t,u)}{(1-q^2t^2)(1-q^2u^2)(1-q^3t^2u^2)}$$

$$\Big(:= \sum_{0\le D\in E_1+\mathrm{Div}_{P,F}(C-S)+2\,\mathrm{Div}(C-S)} L(s,\chi_D;C-S-S_D)a(s,\mathrm{id},D)|D|^{-w}\Big)$$

$$Z(s,w;\mathrm{id},\delta_{E_0+E_1}) = \frac{f_{E_0+E_1}(t,u)}{(1-q^2t^2)(1-q^2u^2)(1-q^3t^2u^2)}$$

$$\Big(:= \sum_{0\le D\in E_0+E_1+\mathrm{Div}_{P,F}(C-S)+2\,\mathrm{Div}(C-S)} L(s,\chi_D;C-S-S_D)a(s,\mathrm{id},D)|D|^{-}$$

where $f_x(t,u)=\sum_{i,j=0}^{8} a_{i,j,x}t^iu^j$ with $a_{i,j,x}\in\mathbb{Z}$ for $k=0,...,3$ and

$$f(t,u)=\sum_{x\in X(F)} f_x(t,u).$$

We can write the first functional equations as follows:

$$Z(s,w;\mathrm{id},\delta_0) = (\frac{1-t}{1-(qt)^{-1}})^2 Z(1-s, s+w-\frac{1}{2}; \mathrm{id}, \delta_0),$$

$$Z(s,w;\mathrm{id},\delta_{E_0}) = \sqrt{q}t\frac{1-t}{1-(qt)^{-1}} Z(1-s, s+w-\frac{1}{2}; \mathrm{id}, \delta_{E_0}),$$

$$Z(s,w;\mathrm{id},\delta_{E_1}) = qt^2 Z(1-s, s+w-\frac{1}{2}; \mathrm{id}, \delta_{E_1})$$

$$Z(s,w;\mathrm{id},\delta_{E_0+E_1}) = \sqrt{q}t\frac{1-t}{1-(qt)^{-1}} Z(1-s, s+w-\frac{1}{2}; \mathrm{id}, \delta_{E_0+E_1}),$$

By lemma in last section, $\alpha(E,E') = 1$ for all $E, E' \in X(F)$. This gives the second functional equation

$$Z(s,w;\mathrm{id},\mathrm{id}) = Z(w,s;\mathrm{id},\mathrm{id}).$$

Applying the functional equations, the relation with L-function, and computation on sums of quadratic characters, we get

$$Z(s,w;\mathrm{id},\mathrm{id}) = \frac{f(t,u)}{(5t-1)(5u-1)\left(125t^2u^2-1\right)}$$

where

$$\begin{aligned} f(t,u) = {} & 637125t^7u^7 - 252425t^7u^6 + 78125t^7u^5 - 15625t^7u^4 \\ & - 252425t^6u^7 + 175485t^6u^6 - 35625t^6u^5 + 28125t^6u^4 \\ & - 5000t^6u^3 + 78125t^5u^7 - 35625t^5u^6 - 41125t^5u^5 \\ & - 7375t^5u^4 + 4500t^5u^3 + 100t^5u^2 - 15625t^4u^7 + 28125t^4u^6 \\ & - 7375t^4u^5 - 1425t^4u^4 - 4020t^4u^3 + 820t^4u^2 - 5000t^3u^6 \\ & + 4500t^3u^5 - 4020t^3u^4 + 3189t^3u^3 - 669t^3u^2 - 25t^3u + 5t^3 \\ & + 100t^2u^5 + 820t^2u^4 - 669t^2u^3 + 137t^2u^2 + 45t^2u \\ & - 9t^2 - 25tu^3 + 45tu^2 - 25tu + 5t + 5u^3 - 9u^2 + 5u - 1. \end{aligned}$$